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PLANE TRIGONOMETRY

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FIRST BOOK

ON

PLANE TRIGONOMETRY,

COMPRISING

GEOMETRICAL TRIGONOMETRY,

AND ITS APPLICATION TO SURVEYING,

WITH NUMEROUS EXAMPLES.

FOR THE USE OF SCHOOLS.

BY

G. W. HEMMING, M.A.

FELLOW OF ST. JOHN'S COLLEGE, CAMBRIDGE, AND AUTHOR OF AN
ELEMENTARY TREATISE ON THE DIFFERENTIAL AND INTEGRAL CALCULUS

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PREFACE.

PLANE TRIGONOMETRY consists of two parts: the one which may be called Geometrical Trigonometry, having for its object the investigation of the relations between the sides and angles of rectilinear figures, in which the idea of an angle is confined within its geometrical limits of zero and two right angles; the other, Analytical Trigonometry, which extends its investigations to the properties of angles understood in a much wider sense.

Geometrical Trigonometry presents little difficulty to a student who is familiar with Euclid and a few of the simpler propositions of algebra, and is all that is required for the purposes of surveying; while Analytical Trigonometry is encumbered with greater difficulties, and admits of no *immediate* applications of practical interest.

For these reasons it appears desirable to keep the two branches of the subject entirely distinct, and a further reason may be found in the fact that many of the demonstrations commonly employed are applicable only to the former; and that in Treatises where the two are considered together, the truth of such results, when the definitions of Analytical Trigonometry are used, is either tacitly assumed or proved by a number of separate processes, which are unavoidably troublesome and inelegant; while by the separate treatment of the two parts of the subject, the proofs are made co-extensive with the definitions.

By confining the present work strictly to Geometrical Trigonometry, a very interesting subject is placed more within the reach of those who have but little time to devote to such studies, while even those who intend to carry their reading further, will, it is thought, find their advantage in first studying the simpler branch of the subject before coping with the more complex ideas of Analytical Trigonometry.

In order to complete the subject of Plane Trigonometry, a short Treatise on Analytical Trigonometry, with the methods of calculating logarithmic and trigonometrical tables, is in course of preparation.

CONTENTS.

CHAPTER I.

ON THE TRIGONOMETRICAL FUNCTIONS OF ONE ANGLE.

PAGE
1—2. On the representation of lines and angles . . . 1
3—4. Definitions of trigonometrical ratios 3
5—7. On supplementary and complementary angles . . 6
8. To trace the changes in the values of the several trigonometrical functions as the angle varies from zero to two right angles 12
9. To find the trigonometrical functions of 30°, 60° and 45° . . . . —
10—14. To find the relations between the different trigonometrical functions 15
15. Definitions of $\sin^{-1} n$, &c. 20
16. Recapitulation . . . . —
17. Examples 21

CHAPTER II.

OF THE TRIGONOMETRICAL FUNCTIONS OF MORE THAN ONE ANGLE.

18—20. To find the sine and cosine of an angle equal to the sum or difference of two others 23
21—23. To express the sine and cosine of $2A$ in terms of those of A and the converse 30

PAGE
24—25. To express the products of two sines or cosines in terms of the sums or differences of sines and cosines and the converse 32
26. To express the tangent of the sum or difference of two angles in terms of those of the angles themselves 34
27. To express sin $3A$ and cos $3A$ in terms of sin A and cos A 35
28. To determine the sines and cosines of 18^0, 9^0, 27^0, 36^0, 54^0, 63^0, 72^0 and 81^0 . . . . . . —
29. Recapitulation 37
30. Examples 38

CHAPTER III.

ON THE NATURE AND USE OF TABLES OF THE LOGARITHMS OF NUMBERS AND TRIGONOMETRICAL FUNCTIONS.

31. Definition of a logarithm 49
32. To perform by means of logarithms the operations of multiplication, division, involution and evolution, by addition, subtraction, multiplication and division . . . . —
33—36. $\log_a a = 1$. $\log_a 1 = 0$. The logarithm of a fraction to an integral base is negative, and that of a negative number to a positive base impossible . . 51
37. To trace the values of logarithms of numbers to the base 10 52
38—39. The mantissæ of the logarithms of all numbers having the same significant digits, are the same . 54
40—42. On the mode of using tables of the logarithms of numbers 55
43. Examples 63
44—46. On the mode of using tables of the logarithms of trigonometrical functions 67
47—48. On the choice of logarithmic formulæ . . . 74
49. On the use of subsidiary angles 79
50. Examples 80

CHAPTER IV.

ON THE SOLUTION OF TRIANGLES.

PAGE
51. When three parts of a triangle are given, it can generally be solved 83
52—53. The sides of a triangle are proportional to the sines of the opposite angles 84
54. To express the cosine of an angle of a triangle in terms of the sides 85
56. On the solution of right-angled triangles . . 88
57. On the solution of oblique-angled triangles . . 90
58. Examples 100

CHAPTER V.

ON SURVEYING.

59—65. On the instruments used in surveying . . . 105
66. Examples of the determination of heights and distances 111

CHAPTER VI.

ON THE AREAS OF TRIANGLES AND POLYGONS, AND THE RADII OF THEIR INSCRIBED AND CIRCUMSCRIBED CIRCLES.

67. To determine the area of a triangle . . . 128
68. To find the radius of the circle described about a triangle 129
69. To find the radius of the circle inscribed in a triangle 130
70. To find the area of a regular polygon and the radii of the inscribed and circumscribed circles . . 131
71. Examples 132

GEOMETRICAL TRIGONOMETRY.

CHAPTER I.

ON THE TRIGONOMETRICAL FUNCTIONS OF ONE ANGLE.

1. THE object of Geometrical Trigonometry is the investigation of the relations which exist among the sides and angles of rectilinear figures. This purpose is greatly aided by departing from strict geometrical methods such as those employed by Euclid, and adopting numerical measures of lines and angles. To express numerically the length of a line, we have only to fix upon some standard of length to be denoted by unity, and then every other line will be denoted by the number of such units, and fractions of such units, that it contains. Thus, if a foot be chosen for the unit of length, a line 10 feet long must be represented by the number 10, and one 15 feet long by the number 15. If a yard had been our unit of length, the same

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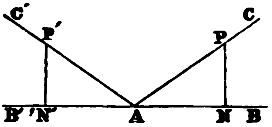
ed into 60 parts, called minutes, and each te into 60 parts, called seconds; and all angles then described by stating the number of degrees, nutes, and seconds which they contain. Any aller portion of an angle is represented by a raction or decimal of a second, but angles less than one second may generally be neglected in practice. An angle of a degrees, b minutes, and c seconds, is written a°, b', c''.

3. *Definitions of trigonometrical ratios.*



Let BAC be any *acute* angle. From any point P in AC draw PN perpendicular to AB.

Then the ratio $\dfrac{PN}{AP}$ is called the sine of BAC,

and the ratio $\dfrac{AN}{AP}$ is called the cosine of BAC.

Again, let BAC' be any *obtuse* angle. Produce BA to B', and from any point P' in AC' draw $P'N'$ perpendicular to BA *produced.*

Then $\dfrac{P'N'}{AP'}$ is called the sine of BAC'

and $-\dfrac{AN'}{AP'}$ is called the cosine of B

---

* The reader may perhaps be surprised to find

If we denote the angle *BAC* by *A*, and *BAC′* by *A′*, the above definitions may be concisely expressed as follows:

$$\sin A = \frac{PN}{AP} \qquad \sin A' = \frac{P'N'}{AP'}$$

$$\cos A = \frac{AN}{AP} \qquad \cos A' = -\frac{AN'}{AP'}$$

The following definitions are also important:

The ratio of the sine to the cosine of any angle is called the tangent of the angle.

The reciprocal of the sine is called the cosecant.

The reciprocal of the cosine is called the secant.

The reciprocal of the tangent is called the cotangent.

Hence we have, referring to the figure and using the abbreviated form commonly adopted,

$$\tan A = \frac{\sin A}{\cos A} = \frac{\frac{PN}{AP}}{\frac{AN}{AP}} = \frac{PN}{AN}$$

---

positive ratio is denoted by the sine of any angle, whether acute or obtuse, the cosine is defined by a positive ratio in the former, and by a negative one in the latter case. At present, he must be content to take it as a mere matter of definition. The reasons which render such a definition desirable will be explained in Analytical Trigonometry.

$$\tan A' = \frac{\sin A'}{\cos A'} = \frac{\dfrac{P'N'}{AP'}}{-\dfrac{AN'}{AP'}} = -\frac{P'N'}{AN'}$$

$$\operatorname{cosec} A = \frac{1}{\sin A} = \frac{AP}{PN}$$

$$\operatorname{cosec} A' = \frac{1}{\sin A'} = \frac{AP'}{P'N'}$$

$$\sec A = \frac{1}{\cos A} = \frac{AP}{AN}$$

$$\sec A' = \frac{1}{\cos A'} = -\frac{AP'}{AN'}$$

$$\cot A = \frac{1}{\tan A} = \frac{AN}{PN}$$

$$\cot A' = \frac{1}{\tan A'} = -\frac{AN'}{P'N'}$$

These geometrical values of the several ratios are sometimes taken as their definitions.

Two other definitions are generally given, viz.:

versine $A = 1 - \cos A$ A being any angle,
suversine $A = 1 + \cos A$.

These are however of very little use, as the forms $1 - \cos A$ and $1 + \cos A$ may be used as conveniently as the others.

It will be observed that the sine and cosecant of an angle are always positive, but that the cosine, tangent, secant and cotangent are positive or negative, according as the angle is acute or obtuse.

4. All the quantities above defined are expressed in terms of the ratio of certain lines, and are therefore numerical quantities, being equal to the fractions whose numerators and denominators are proportional to the number of units of length in the respective lines. The values of these fractions are evidently unaltered by any change in the unit of length employed, since that would change their numerators and denominators in the same proportion. They are also independent of the position of the points P, P', from which the perpendiculars are dropped, since if we took any other points in AC and AC', the triangles formed would be similar to ANP and $AN'P'$, and the ratios of their homologous sides therefore the same. So also if the perpendiculars had been dropped from points in AB and AB' upon AC and AC', the ratios would have remained the same. The values of the trigonometrical ratios depend therefore on that of the angle, and on that alone. For this reason they are called *trigonometrical functions* of the angle, the function of any quantity being a term used to describe any other quantity whose value depends upon that of the former.

5. Two angles, which together make up two right angles, or 180°, are called *supplementary* angles,

and each of them is called the ***supplement*** of the other.

The angles BAC, $B'AC$ in the figure of Art. 3 are therefore supplementary angles, *i. e.* $B'AC = 180^\circ - BAC$. Now by the definitions, since $B'AC$ is an obtuse angle,

$$\sin B'AC = \frac{PN}{AP} = \sin BAC$$

$$\cos B'AC = -\frac{AN}{AP} = -\cos BAC$$

$$\tan B'AC = \frac{\sin B'AC}{\cos B'AC} = \frac{\sin BAC}{-\cos BAC} = -\tan BAC$$

$$\operatorname{cosec} B'AC = \frac{1}{\sin B'AC} = \frac{1}{\sin BAC} = \operatorname{cosec} BAC$$

$$\sec B'AC = \frac{1}{\cos B'AC} = \frac{1}{-\cos BAC} = -\sec BAC$$

$$\cot B'AC = \frac{1}{\tan B'AC} = \frac{1}{-\tan BAC} = -\cot BAC$$

or

$$\begin{aligned}
\sin(180^\circ - A) &= \sin A \\
\cos(180^\circ - A) &= -\cos A \\
\tan(180^\circ - A) &= -\tan A \\
\operatorname{cosec}(180^\circ - A) &= \operatorname{cosec} A \\
\sec(180^\circ - A) &= -\sec A \\
\cot(180^\circ - A) &= -\cot A
\end{aligned}$$

The sines and cosecants of supplementary angles are therefore equal, and the cosines, tangents, secants and cotangents equal in magnitude, but opposite in sign.

6. Two angles which together make up one right angle, or 90°, are called *complementary* angles, and each is called the *complement* of the other.

In the figure of Art. 3, the three angles of the triangle APN make up two right angles, (Euc. I. 32) and therefore, since ANP is a right angle, BAC and APN are complementary angles, *i. e.* if $BAC=A$, $APN=90^\circ-A$.

Then, from the definitions,

$$\sin APN = \frac{AN}{AP} = \cos BAC$$

or

$$\sin(90^\circ-A)=\frac{AN}{AP}=\cos A$$

similarly

$$\cos(90^\circ-A)=\frac{NP}{AP}=\sin A$$

and $\tan(90^\circ-A)=\dfrac{\sin(90^\circ-A)}{\cos(90^\circ-A)}=\dfrac{AN}{NP}=\cot A$

and by equating the reciprocals of the above quantities,

$$\begin{aligned}\operatorname{cosec}(90^\circ-A)&=\sec A\\ \sec(90^\circ-A)&=\operatorname{cosec} A\\ \cot(90^\circ-A)&=\tan A\end{aligned}$$

that is, the cosine, cotangent and cosecant of an angle are respectively equal to the sine, tangent and secant of its complement. It is from this property that the former terms are derived.

7. The following results, which are immediately deducible from the two preceding articles, will be found convenient.

$$\begin{aligned} \sin(90^\circ + A) &= \sin(180^\circ - \overline{90^\circ - A}) \\ &= \sin(90^\circ - A) && \text{by Art. 5.} \\ &= \cos A && \text{by Art. 6.} \end{aligned}$$

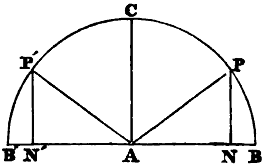
Similarly

$$\cos(90^\circ + A) = -\cos(90^\circ - A) = -\sin A$$
$$\tan(90^\circ + A) = -\tan(90^\circ - A) = -\cot A$$

and equating the reciprocals of these equations

$$\begin{aligned} \operatorname{cosec}(90^\circ + A) &= \sec A \\ \sec(90^\circ + A) &= -\operatorname{cosec} A \\ \cot(90^\circ + A) &= -\tan A \end{aligned}$$

8. *To trace the values of the trigonometrical functions as the angle varies from zero to* 180°.



Let BCB' be a semicircle described about A as centre, BAP any angle, PN perpendicular to AB, and AC perpendicular to AB. If we suppose the angle BAP to assume in succession all values from zero to 90°, the point P must move along the arc BC from

B to *C*. *PN* therefore increases from zero to *AC* or *AP*, and $\frac{PN}{AP}$ or sin *BAP* increases from 0 to 1.

As the angle *BAP'* increases from 90° to 180°, *P'* moves from *C* to *B'*, *P'N'* evidently diminishes from *AC* or *AP'* to zero, and the sine, or $\frac{P'N'}{AP'}$, diminishes from 1 to 0.

The values of the sine therefore occur in pairs, the two angles which have the same sine, being the one acute and the other obtuse, and, as was shewn in Art. 5, supplementary to one another.

Again, as the angle increases from zero to 90°, *AN* diminishes from *AB* or *AP* to zero, and therefore the cosine, or $\frac{AN}{AP}$, diminishes from 1 to 0.

As the angle increases from 90° to 180°, *AN'* increases from zero to *AB'* or *AP'*, and the cosine, or $-\frac{AN'}{AP'}$, diminishes from 0 to -1.

Hence the values of the cosine do not recur like those of the sine, since those cosines whose magnitudes are the same have opposite signs.

Since the tangent equals the sine divided by the cosine; as the angle increases from zero to 90°, the tangent increases from $\frac{0}{1}$ to $\frac{1}{0}$, or from 0 to ∞, and as the angle increases from 90° to 180°, the tangent, which is now negative, varies from $-\infty$ to 0.

So also the cosecant, which is the reciprocal of the sine, varies from ∞ to 1, as the angle increases from zero to 90°, and from 1 to ∞ as it increases from 90° to 180°.

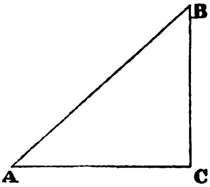
The secant, or reciprocal of the cosine, varies from 1 to ∞ as the angle increases from zero to 90°, and from $-\infty$ to -1 as it increases from 90° to 180°.

The cotangent, or reciprocal of the tangent, varies from ∞ to 0 as the angle increases from zero to 90°, and from 0 to $-\infty$ as it increases from 90° to 180°.

It will be observed that the values of the cosecant, like those of the sign, occur in pairs (corresponding to supplementary angles), but that the values of the other functions never recur both in magnitude and sign. If, therefore, the value of the sine or cosecant of an angle is given to determine the angle, there are two angles supplementary to each other, and, therefore, one acute and the other obtuse, either of which will satisfy the condition, but if any of the other trigonometrical functions is given, the angle is absolutely determined, or in other words, there is an ambiguity in the determination of angles by their sines or cosecants, which is avoided by the use of the other trigonometrical functions. The importance of this observation will be seen in the chapter on the solution of triangles.

9. The general problem to determine the values of the sine, cosine, &c., of any given angle is one of considerable difficulty; there are, however, some angles whose sines, cosines and tangents may be readily found by particular methods. Thus the sines, cosines and tangents of the angles 45°, 60° and 30° may be found as follows:

I. *To find the sine, cosine and tangent of* 45°,



Let ABC be a triangle right angled at C, and having the sides AC and BC equal.

Then the angles A and B will also be equal (Euc. I. 5); and since the three angles A, B and C are together equal to two right angles (Euc. I. 32), of which C is one right angle; the remaining angles A and B must each be equal to half a right angle, or 45°.

$$\text{Now} \qquad AB^2 = AC^2 + BC^2 \qquad \text{Euc. I. 49.}$$

$$= 2BC^2$$

$$\therefore \quad AB = \sqrt{2}BC = \sqrt{2}AC$$

$$\therefore \quad \sin 45° = \frac{BC}{AB} = \frac{BC}{\sqrt{2}BC} = \frac{1}{\sqrt{2}}$$

$$\cos 45° = \frac{AC}{AB} = \frac{AC}{\sqrt{2}AC} = \frac{1}{\sqrt{2}}$$

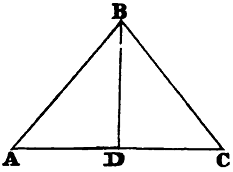
$$\tan 45° = \frac{BC}{AC} = 1$$

The reciprocals of the above equations give

$$\operatorname{cosec} 45° = \sqrt{2}$$
$$\sec 45° = \sqrt{2}$$
$$\cot 45° = 1$$

II. *To find the sines, cosines and tangents of* 60° *and* 30°.

Let ABC be an equilateral triangle.



Then the three angles A, B and C of this triangle are equal, (Euc. I. 5) and therefore each of them is equal to one-third of two righ angles, or 60°.

Draw BD perpendicular to AC.

Then in the triangles ABD and BDC

$$AB = BC$$
$$\angle BAD = \angle BCD$$

and BD is common,

$\therefore$ the triangles are equal, and

$$\angle ABD = \angle CBD$$

and $AD = DC$.

Hence $\angle ABD = \frac{1}{2} ABC = 30°$

and $$AD=\frac{1}{2}AC=\frac{1}{2}AB$$

also $$BD^2=AB^2-AD^2 \qquad \text{Euc. I. 49.}$$

$$=AB^2-\frac{1}{4}AB^2=\frac{3}{4}AB^2$$

$$\therefore\ BD=\frac{\sqrt{3}}{2}AB$$

$$\therefore\ \sin 60^\circ=\frac{BD}{AB}=\frac{\sqrt{3}}{2}$$

$$\cos 60^\circ=\frac{AD}{AB}=\frac{1}{2}$$

$$\tan 60^\circ=\frac{BD}{AD}=\frac{\frac{\sqrt{3}}{2}AB}{\frac{1}{2}AB}=\sqrt{3}.$$

The reciprocals of the above equations give

$$\operatorname{cosec} 60^\circ=\frac{2}{\sqrt{3}}$$

$$\sec 60^\circ=2$$

$$\cot 60^\circ=\frac{1}{\sqrt{3}}.$$

Hence, since 30° is the complement of 60°

$$\sin 30^\circ=\cos 60^\circ=\frac{1}{2}$$

$$\cos 30^\circ=\sin 60^\circ=\frac{\sqrt{3}}{2}$$

$$\tan 30^\circ=\cot 60^\circ=\frac{1}{\sqrt{3}}$$

and the remaining functions will equal the respective reciprocals of the above numbers.

Having found the values of the trigonometrical functions of 30°, 45° and 60°, those of 150°, 135° and 120° are known at once by means of Art. 5.

Thus

$$\sin 150^\circ = \sin (180^\circ - 150^\circ) = \sin 30^\circ = \frac{1}{2}$$

$$\cos 135^\circ = -\cos (180^\circ - 135^\circ) = -\cos 45^\circ = -\frac{1}{\sqrt{2}}$$

$$\tan 120^\circ = -\tan (180^\circ - 120^\circ) = -\tan 60^\circ = -\sqrt{3}$$

and the other functions may be similarly determined.

10. When the value of any one of the trigonometrical functions is given, that of the angle, and thence the values, of the remaining functions are determinable subject to the ambiguity already mentioned when the given function is a sine or cosecant. Each of these functions must therefore be capable of being expressed in terms of any of the others. We shall now seek the equations by which the several functions are connected, as follows:

Referring to the figure of Art. 3, we have

$$PN^2 + AN^2 = AP^2 \qquad \text{Euc. I. 49}$$

$$\therefore \frac{PN^2}{AP^2} + \frac{AN^2}{AP^2} = 1.$$

But $\frac{PN}{AP}=\sin BAC \quad \frac{AN}{AP}=\cos BAC$

$$\therefore \quad (\sin BAC)^2+(\cos BAC)^2=1$$

Again, $\quad P'N'^2+AN'^2=AP'^2$

$$\therefore \quad \frac{P'N'^2}{AP'^2}+\frac{AN'^2}{AP'^2}=1$$

and $\quad \frac{P'N'}{AP'}=\sin BAC' \quad \frac{AN'}{AP'}=-\cos BAC'$

$$\therefore \quad (\sin BAC')^2+(\cos BAC')^2=1$$

the same formula as that above obtained for an acute angle. Hence, if A be any angle,

$$\sin^2 A+\cos^2 A=1 \quad . \quad . \quad . \quad . \quad (1)$$

$\sin^2 A$, $\cos^2 A$ being the usual abbreviations for $(\sin A)^2$ and $(\cos A)^2$.

We have also from the definitions (A being any angle)

$$\tan A=\frac{\sin A}{\cos A} \quad . \quad . \quad . \quad . \quad (2)$$

$$\operatorname{cosec} A=\frac{1}{\sin A} \quad . \quad . \quad . \quad . \quad (3)$$

$$\sec A=\frac{1}{\cos A} \quad . \quad . \quad . \quad . \quad (4)$$

$$\cot A=\frac{1}{\tan A} \quad . \quad . \quad . \quad . \quad (5)$$

From these five equations we may eliminate any four of the six functions involved, and obtain an equation connecting the remaining two.

11. *To express each of the other functions explicitly in terms of the sine.*

We have

from (1) $\cos A = \pm\sqrt{1-\sin^2 A}$

from (2) $\tan A = \dfrac{\sin A}{\cos A} = \pm\dfrac{\sin A}{\sqrt{1-\sin^2 A}}$

from (3) $\operatorname{cosec} A = \dfrac{1}{\sin A}$

from (4) $\sec A = \dfrac{1}{\cos A} = \pm\dfrac{1}{\sqrt{1-\sin^2 A}}$

from (5) $\cot A = \dfrac{1}{\tan A} = \pm\dfrac{\sqrt{1-\sin^2 A}}{\sin A}$

by which all the remaining functions are given explicitly in terms of the sine.

The double sign in the values of the cosine, secant, tangent and cotangent corresponds to the two supplementary angles, which have the same sine, but whose cosines, secants, tangents and cotangents have opposite algebraical signs.

Thus suppose we have the equation $\sin A = \dfrac{\sqrt{3}}{2}$ given to determine the value of $\cos A$.

$$\cos A = \pm\sqrt{1-\frac{3}{4}} = \pm\frac{1}{2}$$

that is, the value of cos A is either $+\frac{1}{2}$ or $-\frac{1}{2}$, the former corresponding to the acute value of A, which satisfies the given equation (and which we have found in Art. 9 to be 60°), and the latter to the obtuse value 120°, which is supplementary to the former, and also satisfies the equation $\sin A = \frac{\sqrt{3}}{2}$

12. *To express each of the other functions explicitly in terms of the cosine.*

This is done by the following equations, which are immediately deducible from the five equations of Art. 10.

$$\sin A = \sqrt{1-\cos^2 A}$$

the positive sign of the root being taken because sin A cannot be negative.

$$\tan A = \frac{\sin A}{\cos A} = \frac{\sqrt{1-\cos^2 A}}{\cos A}$$

$$\sec A = \frac{1}{\cos A}$$

$$\operatorname{cosec} A = \frac{1}{\sin A} = \frac{1}{\sqrt{1-\cos^2 A}}$$

$$\cot A = \frac{1}{\tan A} = \frac{\cos A}{\sqrt{1-\cos^2 A}}$$

13 *To express each of the other functions explicitly in the terms of the tangent.*

By dividing equation (1) of Art. 10 by $\cos^2 A$ we obtain

$$1+\tan^2 A=\sec^2 A$$

$$\therefore \quad \sec A=\pm\sqrt{1+\tan^2 A}$$

The positive or negative sign of the root must be chosen according as the given value of $\tan A$ is positive or negative, since the tangent and secant of an angle always have the same sign, being both positive if the angle is acute, and both negative if it is obtuse. From the above equation we obtain

$$\cos A=\frac{1}{\sec A}=\frac{\pm 1}{\sqrt{1+\tan^2 A}}$$

$$\sin A=\tan A \cdot \cos A=\frac{\pm\tan A}{\sqrt{1+\tan^2 A}}$$

$$\operatorname{cosec} A=\frac{1}{\sin A}=\pm\frac{\sqrt{1+\tan^2 A}}{\tan A}$$

$$\cot A=\frac{1}{\tan A}$$

14. The values of the several trigonometrical functions in terms of the cosecant, secant and cotangent may be found from the above by putting $\frac{1}{\operatorname{cosec} A}$ for $\sin A$ in the equations of Art. 11, $\frac{1}{\sec A}$ for $\cos A$ in those of Art. 12, and $\frac{1}{\cot A}$ for $\tan A$ in those of Art 13. Thus, for example, to express $\cot A$ in

terms of cosec A, we must change sin A into $\dfrac{1}{\operatorname{cosec} A}$ in the last equation of Art. 11, when we obtain

$$\cot A = \pm \operatorname{cosec} A \sqrt{1 - \frac{1}{\operatorname{cosec}^2 A}} = \pm \sqrt{\operatorname{cosec}^2 A - 1}$$

which result may also be obtained at once by dividing equation (1) of Art. 10 by $\sin^2 A$, when it becomes

$$1 + \cot^2 A = \operatorname{cosec}^2 A$$

15. An angle whose sine equals n is frequently denoted by $\sin^{-1} n$, and a similar notation is used with reference to the other trigonometrical functions.

Thus, for example, $\sin^{-1} \dfrac{\sqrt{3}}{2} = 60^\circ$

$$\tan^{-1} 1 = 45^\circ$$

$$\cot^{-1} \sqrt{3} = 30^\circ$$

16. Some of the most important of the results obtained in this chapter are here collected, and should be committed to memory.

$$\begin{aligned}
\sin A &= \sin(180^\circ - A) = \cos(90^\circ - A) = -\cos(90^\circ + A) \\
\cos A &= -\cos(180^\circ - A) = \sin(90^\circ - A) = \sin(90^\circ + A) \\
\tan A &= -\tan(180^\circ - A) = \cot(90^\circ - A) = -\cot(90^\circ + A)
\end{aligned}$$

$$\sin^2 A + \cos^2 A = 1$$

$$\tan A = \frac{\sin A}{\cos A}$$

$$\cot A=\frac{1}{\tan A}=\frac{\cos A}{\sin A}$$

$$\sec A=\frac{1}{\cos A}$$

$$\text{cosec } A=\frac{1}{\sin A}$$

$$\sin A= \quad \sqrt{1-\cos^2 A}=\pm\frac{\tan A}{\sqrt{1+\tan^2 A}}$$

$$\cos A=\pm\sqrt{1-\sin^2 A}=\pm\frac{1}{\sqrt{1+\tan^2 A}}$$

$$\sec^2 A=1+\tan^2 A$$

$$\text{cosec}^2 A=1+\cot^2 A$$

17. *Examples:*

1. $\sin A+\cos A=\sqrt{2}$ Find A.

$$\sin A=\sqrt{2}-\cos A$$

$$\therefore\quad 1-\cos^2 A=2-2\sqrt{2}\cos A+\cos^2 A$$

$$\therefore\quad 2\cos^2 A-2\sqrt{2}\cos A+1=0$$

$$\therefore\quad \cos^2 A-\sqrt{2}\cos A+\frac{1}{2}=0$$

$$\therefore\quad \cos A=\frac{1}{\sqrt{2}}$$

$$\therefore\quad A=45^\circ$$

2. $\tan A+\cot A=\frac{4}{\sqrt{3}}$ Find A.

By multiplying by $\tan A$, we obtain a quadratic equation in $\tan A$, which gives $A=30^\circ$ or 60°.

3. $\sin A - \cos A = 1$ $\quad A = 90^\circ$ or 180°

4. $\sin^2 A - \cos^2 A = \frac{1}{2}$ $\quad A = 60^\circ$ or 120°

5. $\operatorname{cosec} A = \sqrt{3} \sec A$ $\quad A = 30^\circ$

6. $\tan A = 3 \cot A$ $\quad A = 60^\circ$ or 120°

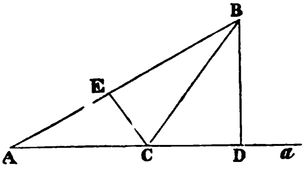
7. $\tan^2 A + 5 \cot^2 A + 2 = 4 \operatorname{cosec}^2 A$ $\quad A = 45^\circ$ or 135°

CHAPTER II.

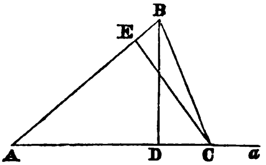
ON THE TRIGONOMETRICAL FUNCTIONS OF MORE THAN ONE ANGLE.

18. *To find the sine and cosine of an angle equal to the sum of two given angles.*

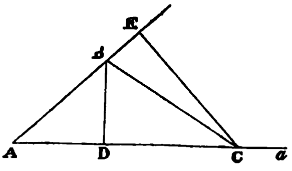
I.



II.



III.



Let A and B be the two given angles. Construct a triangle ABC, having the angle BAC equal to A, and the angle ABC equal to B. Then if AC is produced to a, the exterior angle BCa will equal $A+B$, (Euc. I. 32.) From B draw BD perpendicular to AC, or AC produced, and from C draw CE perpendicular to AB, or AB produced.

There are three distinct cases to be considered:

1. Where A, B and $A+B$ are all acute angles, as in Fig. I.

2. Where A and B are acute, and $A+B$ obtuse, as in Fig. II.

3. Where one of the angles A and B is acute, and the other obtuse, as in Fig. III, where A is acute and B obtuse, in which case $A+B$ is necessarily an obtuse angle.

The case where A and B are both obtuse is impossible, since $A+B$ would then exceed 180°.

In either of the first two cases, we have, by equating two values of sin A,

$$\frac{BD}{AB}=\frac{CE}{AC}$$

$$\therefore \quad BD=\frac{CE}{AC}(BE+AE)$$

$$\therefore \quad \frac{BD}{BC}=\frac{CE}{AC}\cdot\frac{BE}{BC}+\frac{AE}{AC}\cdot\frac{CE}{BC}$$

But in Figs. I. and II. $\frac{BD}{BC}=\sin\,(A+B)$

$$\frac{CE}{AC}=\sin A,\ \frac{BE}{BC}=\cos B,\ \frac{AE}{AC}=\cos A,\ \frac{CE}{BC}=\sin B$$

$$\therefore\ \sin\,(A+B)=\sin A\cos B+\cos A\sin B.$$

In the third case we have (with Fig. III.)

$$\frac{BD}{AB}=\frac{CE}{AC}$$

$$\therefore \quad BD\,.\,AC=CE\,.\,(AE-BE)$$

$$\frac{BD}{BC}=\frac{CE}{BC}\cdot\frac{AE}{AC}-\frac{CE}{AC}\cdot\frac{BE}{BC}$$

and $$\frac{BD}{BC}=\sin\,(A+B)$$

$$\frac{CE}{BC}=\sin B \qquad \frac{AE}{AC}=\cos A \qquad \frac{CE}{AC}=\sin A$$

$$\frac{BE}{BC}=\cos CBE=-\cos B$$

$$\therefore\ \sin\,(A+B)=\sin A\cos B+\cos A\sin B$$

the same formula as in the other cases.

Again, to determine $\cos\,(A+B)$, we have in the first case, by equating two values of $\cos A$,

$$\frac{AD}{AB}=\frac{AE}{AC}$$

$$\therefore\quad AC\,.\,(AC+CD)=AE(AE+BE)$$

$$\therefore\quad AC\,.\,CD=AE\,.\,BE-CE^2$$

since $$AC^2-AE^2=CE^2$$

$$\therefore\ \frac{CD}{BC}=\frac{AE}{AC}\cdot\frac{BE}{BC}-\frac{CE}{AC}\cdot\frac{CE}{BC}$$

and $\frac{CD}{BC}=\cos\,(A+B)$ $\frac{AE}{AC}=\cos A$ $\frac{BE}{BC}=\cos B$

$$\frac{CE}{AC}=\sin A \quad \frac{CE}{BC}=\sin B$$

$$\therefore\ \cos\,(A+B)=\cos A\cos B-\sin A\sin B.$$

The reasoning in the second and third cases is precisely the same, except that in Figs. II. and III.

$$AD=AC-CD \text{ instead of } AC+CD$$

and $\frac{CD}{BC} = \cos(BCA) = -\cos(A+B)$ instead of $\cos(A+B)$

and in Fig. III. we also have

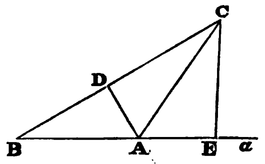
$$AB = AE - BE \text{ instead of } AE + BE.$$

and $\frac{BE}{BC} = \cos CBE = -\cos B$ instead of $\cos B$.

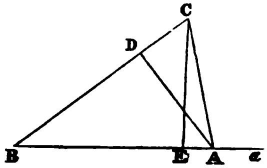
The student should complete the proofs with these alterations, when he will arrive at the same result as in the first case.

19. *To find the sine and cosine of an angle equal to the difference of two given angles.*

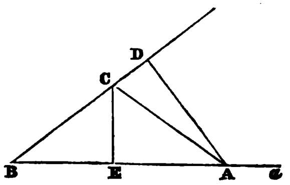
I.



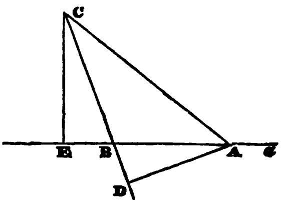
II.



III.



IV.



Let A and B be the two given angles of which A is the greater.

Construct a triangle ABC, and produce BA to a, making the exterior angle CAa equal to A, and the angle ABC equal to B. Then the angle BCA will equal $A-B$. Draw AD perpendicular to BC, or BC produced and CE perpendicular to BA or BA produced.

There are four distinct cases to be considered:

1. Where both A and B are acute, as in Fig. I. in which case $A-B$ must also be acute.
2. Where A is obtuse, and B and $A-B$ acute, as in Fig. II.
3. Where A is obtuse, B acute, and $A-B$ obtuse, as in Fig. III.
4. Where A and B are both obtuse, as in Fig. IV. in which case $A-B$ must be acute.

In the first case we have, by equating two values of tan ABD,

$$\frac{AD}{BD}=\frac{CE}{BE}$$

$$\therefore \quad AD(BA+AE)=CE\,.\,BD$$

$$\therefore \quad AD\,.\,BA=CE\,.\,BD-AE\,.\,AD$$

$$\therefore \quad \frac{AD}{AC}=\frac{CE}{AC}\cdot\frac{BD}{BA}-\frac{AE}{AC}\cdot\frac{AD}{BA}$$

But $\frac{AD}{AC}=\sin(A-B)$ $\frac{CE}{AC}=\sin A$ $\frac{BD}{BA}=\cos B$

$\frac{AE}{AC}=\cos A$ $\frac{AD}{BA}=\sin B$

$$\therefore \quad \sin(A-B)=\sin A\cos B-\cos A\sin B$$

The reasoning in the remaining cases is the same, except that in Figs. II. and III.

$$BE=BA-AE, \text{ instead of } BA+AE,$$

and $\dfrac{AE}{AC}=\cos BAC=-\cos A$, instead of $\cos A$,

and in Fig. IV.

$$BE=AE-BA, \text{ instead of } BA+AE,$$

and $\dfrac{AE}{AC}=\cos BAC=-\cos A$, instead of $\cos A$,

and $\dfrac{BD}{BA}=\cos ABD=-\cos B$, instead of $\cos B$.

On completing the proofs with these changes, the same result will be obtained in every case.

Again, to determine $\cos(A-B)$, we have in the first case, by equating two values of $\cos EBC$,

$$\frac{BE}{BC}=\frac{BD}{BA}$$

$$\therefore \quad BE(BE-AE)=BC(BC-DC)$$

$$\therefore \quad BC\,.\,DC=AE\,.\,BE+CE^2$$

since $\quad BC^2-BE^2=CE^2$

$$\therefore \quad \frac{DC}{AC}=\frac{AE}{AC}\cdot\frac{BE}{BC}+\frac{CE}{AC}\cdot\frac{CE}{BC}$$

But $\dfrac{DC}{AC}=\cos(A-B)$ $\dfrac{AE}{AC}=\cos A$ $\dfrac{BE}{BC}=\cos B$

$\dfrac{CE}{AC}=\sin A$ $\dfrac{CE}{BC}=\sin B$

$$\therefore \quad \cos(A-B)=\cos A\cos B+\sin A\sin B$$

The reasoning in the remaining cases is the same, except that in Figs. II. and III.

$BA = BE + AE$, instead of $BE - AE$

and $\dfrac{AE}{AC} = \cos BAC = -\cos A$, instead of $\cos A$,

and in Fig. III.

$BD = BC + DC$, instead of $BC - DC$,

and $\dfrac{DC}{AC} = \cos ACD = -\cos (A - B)$, instead of $\cos (A - B)$

and in Fig. IV.

$BA = AE - BE$, instead of $BE - AE$

$BD = DC - BC$, instead of $BC - DC$

$\dfrac{AE}{AC} = \cos CAE = -\cos A$, instead of $\cos A$,

and $\dfrac{BE}{BC} = \cos CBE = -\cos B$, instead of $\cos B$.

On completing the proofs with these changes, the same result will be obtained in every case.

20. In the preceding articles, we have supposed the angles under consideration to be always either acute or obtuse, without considering the particular cases where any of them equal 0°, 90°, or 180°. Since, however, the same formulæ have been shewn to hold for all acute and obtuse angles, however near to the values 0°, 90°, or 180°, it is clear that they must be true also when any of the angles have these

particular values. The student may further convince himself of this fact, by giving one of these latter values to any of the angles in the above formulæ, when he will find that the formulæ reduce themselves to self-evident equations, or to some of the simple results obtained in Chapter I. Thus, for example, if in the equation

$$\sin (A+B)=\sin A\,.\cos B+\cos A \sin B,$$

we put $A=90°$ it becomes

$$\sin (90°+B)=\sin 90° \cos B+\cos 90° \sin B$$

or $\cos B=\cos B$, since $\sin 90°=1$, $\cos 90°=0$.

Hence we have for all possible values of the angles A and B,

$$\sin (A+B)=\sin A \cos B+\cos A \sin B$$
$$\sin (A-B)=\sin A \cos B-\cos A \sin B$$
$$\cos (A+B)=\cos A \cos B-\sin A \sin B$$
$$\cos (A-B)=\cos A \cos B+\sin A \sin B$$

21. *To express the sine and cosine of 2A in terms of those of A.*

In the equation

$$\sin (A+B)=\sin A \cos \mathrm{B}+\cos A \sin B,$$

let $B=A$

$$\therefore\ \sin 2A=2 \sin A \cos A$$

Again, in the equation

$$\cos (A+B)=\cos A \cos B-\sin A \sin B,$$

let $B=A$

$$\therefore\ \cos 2A=\cos^2 A-\sin^2 A$$

22. *To express the sine and cosine of A in terms of the cosine of 2A.*

We have $\cos^2 A - \sin^2 A = \cos 2A$

and $\cos^2 A + \sin^2 A = 1$

$$\therefore \quad 2\cos^2 A = 1 + \cos 2A$$

$$2\sin^2 A = 1 - \cos 2A$$

or $\cos A = \sqrt{1+\cos 2A}$

$$\sin A = \sqrt{1-\cos 2A}$$

The positive signs of the roots must be taken, because $2A$ cannot exceed 180°, and, therefore, A must be acute and $\sin A$ and $\cos A$ both positive.

Hence also

$$\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

expressions which are of very frequent occurrence.

23. *To express the sine and cosine of A in terms of the sine of 2A.*

We have $2\sin A \cos A = \sin 2A$

and $\sin^2 A + \cos^2 A = 1$

$\therefore$ adding $(\sin A + \cos A)^2 = 1 + \sin 2A$

and subtracting $(\sin A - \cos A)^2 = 1 - \sin 2A$

or $\sin A + \cos A = \sqrt{1+\sin 2A} \qquad (1)$

$$\sin A - \cos A = \pm\sqrt{1-\sin 2A} \qquad (2)$$

The positive sign of the root is taken in the first equation because $2A$ cannot exceed 180°, and therefore A must be acute, and $\sin A$ and $\cos A$ both positive. In the second equation, the positive or negative sign must be taken according as sin A is greater or less than cos A. Now, when A equals 45°, $\sin A = \cos A$, and an increase in the angle corresponds to an increase in the sine and a decrease in the cosine. If, therefore, A is greater than 45°, sin A is greater than cos A; and if less than 45°, sin A is less than cos A. In the former case, therefore, the positive, and in the latter the negative sign must be used.

Hence by adding equations (1) and (2), we obtain (if A is greater than 45°, *i. e.* if $2A$ is obtuse)

$$2 \sin A = \sqrt{1+\sin 2A} + \sqrt{1-\sin 2A}$$

$$2 \cos A = \sqrt{1+\sin 2A} - \sqrt{1-\sin 2A}$$

or

$$\sin A = \frac{1}{2}\sqrt{1+\sin 2A} + \frac{1}{2}\sqrt{1-\sin 2A}$$

$$\cos A = \frac{1}{2}\sqrt{1+\sin 2A} - \frac{1}{2}\sqrt{1-\sin 2A}$$

If A is less than 45°, *i. e.* if $2A$ is acute, the corresponding equations are

$$\sin A = \frac{1}{2}\sqrt{1+\sin 2A} - \frac{1}{2}\sqrt{1-\sin 2A}$$

$$\cos A = \frac{1}{2}\sqrt{1+\sin 2A} + \frac{1}{2}\sqrt{1-\sin 2A}.$$

24. *To express the products of two sines or cosines in terms of the sums or differences of sines and cosines.*

Since

$$\sin(A+B)=\sin A\cos B+\cos A\sin B$$

and $$\sin(A-B)=\sin A\cos B-\cos A\sin B$$

$$\therefore\quad \sin(A+B)+\sin(A-B)=2\sin A\cos B$$

and $$\sin(A+B)-\sin(A-B)=2\cos A\sin B$$

$$\therefore\quad \sin A\cos B=\frac{1}{2}\left\{\sin(A+B)+\sin(A-B)\right\}$$

and $$\cos A\sin B=\frac{1}{2}\left\{\sin(A+B)-\sin(A-B)\right\}$$

Again from the equations

$$\cos(A+B)=\cos A\cos B-\sin A\sin B$$
$$\cos(A-B)=\cos A\cos B+\sin A\sin B$$

we obtain in a similar manner

$$\cos A\cos B=\frac{1}{2}\left\{\cos(A-B)+\cos(A+B)\right\}$$

$$\sin A\sin B=\frac{1}{2}\left\{\cos(A-B)-\cos(A+B)\right\}$$

25. *To express the sums or differences of two sines or cosines in terms of the products of sines and cosines.*

If A and B be any two angles of which A is the greater.

$$A=\frac{1}{2}(A+B)+\frac{1}{2}(A-B)$$

$$B=\frac{1}{2}(A+B)-\frac{1}{2}(A-B$$

$$\therefore \sin A = \sin\left\{\frac{1}{2}(A+B) + \frac{1}{2}(A-B)\right\}$$

$$\therefore \sin B = \sin\left\{\frac{1}{2}(A+B) - \frac{1}{2}(A-B)\right\}$$

$$\therefore \sin A = \sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B) + \cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$$

$$\sin B = \sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B) - \cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$$

$$\therefore \sin A + \sin B = 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$$

$$\sin A - \sin B = 2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$$

Again $\cos A = \cos\left\{\frac{1}{2}(A+B) + \frac{1}{2}(A-B)\right\}$

$$\cos B = \cos\left\{\frac{1}{2}(A+B) - \frac{1}{2}(A-B)\right\}$$

$$\therefore \cos A = \cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B) - \sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$$

$$\cos B = \cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B) + \sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$$

$$\therefore \cos A + \cos B = 2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$$

$$\cos B - \cos A = 2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$$

26. *To express tan $(A+B)$ and tan $(A-B)$ in terms of tan A and tan B.*

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

$$=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}$$

$$=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

In like manner we may obtain

$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$$

If in the former of these equations we put $B=A$, we have

$$\tan 2A=\frac{2\tan A}{1-\tan^2 A}$$

27. *To express* sin $3A$ *in terms of* sin A *and* cos $3A$ *in terms of* cos A.

$$\begin{aligned}\sin 3A&=\sin(2A+A)\\&=\sin 2A\cos A+\cos 2A\sin A\\&=2\sin A\cos^2 A+(1-2\sin^2 A)\sin A\\&=3\sin A-4\sin^3 A\end{aligned}$$

$$\begin{aligned}\cos 3A&=\cos(2A+A)\\&=\cos 2A\cos A-\sin 2A\sin A\\&=(2\cos^2 A-1)\cos A-2\cos A\sin^2 A\\&=4\cos^3 A-3\cos A.\end{aligned}$$

28. *To determine the sine and cosine of* 18°, *and thence those of* 9°, 27°, 36°, 54°, 63°, 72° *and* 81°.

Let $A=18°$, then $5A=90°$.

$$\therefore\quad \sin 2A=\cos 3A$$

$$\therefore\ 2\sin A\cos A=\cos 2A\cos A-\sin 2A\sin A$$

$$=(1-2\sin^2 A)\cos A-2\sin^2 A.\cos A$$

$$\therefore\quad 2\sin A=1-4\sin^2 A$$

$$\therefore\quad \sin^2 A+\frac{1}{2}\sin A=\frac{1}{4}$$

$$\therefore\quad \sin A=\frac{-1\pm\sqrt{5}}{4}$$

The positive sign of the root must be taken because sin A must be positive.

$$\therefore\quad \sin 18^\circ=\frac{\sqrt{5}-1}{4}=\cos 72^\circ,\text{ since } 72^\circ=90^\circ-18^\circ.$$

$$\therefore\quad \cos 18^\circ=\sqrt{1-\sin^2 18}$$

$$=\sqrt{1-\frac{6-2\sqrt{5}}{16}}$$

$$=\frac{\sqrt{10+2\sqrt{5}}}{4}=\sin 72^\circ$$

$$\cos 36^\circ=1-2\sin^2 18^\circ$$

$$=1-\frac{6-2\sqrt{5}}{8}$$

$$=\frac{1+\sqrt{5}}{4}=\sin 54^\circ$$

$$\sin 36^\circ=\sqrt{1-\cos^2 36^\circ}$$

$$=\sqrt{1-\frac{6+2\sqrt{5}}{16}}$$

$$=\frac{\sqrt{10-2\sqrt{5}}}{4}=\cos 54^\circ$$

Also from Art. 23, since 18° is less than 45°,

$$\sin 9^\circ=\frac{1}{2}\sqrt{1+\sin 18^\circ}-\frac{1}{2}\sqrt{1-\sin 18^\circ}$$

$$\cos 9^\circ=\frac{1}{2}\sqrt{1+\sin 18^\circ}+\frac{1}{2}\sqrt{1-\sin 18^\circ}$$

and $\sqrt{1+\sin 18^\circ}=\sqrt{1+\frac{\sqrt{5}-1}{4}}=\sqrt{\frac{3+\sqrt{5}}{4}}$

$$\sqrt{1-\sin 18^\circ}=\sqrt{1-\frac{\sqrt{5}-1}{4}}=\sqrt{\frac{5-\sqrt{5}}{4}}$$

$$\therefore\ \sin 9^\circ=\frac{\sqrt{3+\sqrt{5}}-\sqrt{5-\sqrt{5}}}{4}=\cos 81^\circ$$

$$\cos 9^\circ=\frac{\sqrt{3+\sqrt{5}}+\sqrt{5-\sqrt{5}}}{4}=\sin 81^\circ$$

In like manner we may obtain, since $27^\circ=\frac{1}{2}.54^\circ$,

$$\sin 27^\circ=\frac{1}{4}\sqrt{5+\sqrt{5}}-\frac{1}{4}\sqrt{3-\sqrt{5}}=\cos 63^\circ$$

$$\cos 27^\circ=\frac{1}{4}\sqrt{5+\sqrt{5}}+\frac{1}{4}\sqrt{3-\sqrt{5}}=\sin 63^\circ$$

29. The most important of the formulæ investigated in this chapter, are here collected, and should be committed to memory.

$$\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$$

$$\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$$

$$\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}$$

$$\sin 2A=2\sin A\cos A$$

$$\begin{aligned}\cos 2A&=\cos^2 A-\sin^2 A\\&=2\cos^2 A-1\\&=1-2\sin^2 A\end{aligned}$$

$$\tan 2A=\frac{2\tan A}{1-\tan^2 A}$$

$$\sin A=\sqrt{\frac{1}{2}(1-\cos 2A)}$$

$$\cos A=\sqrt{\frac{1}{2}(1+\cos 2A)}$$

30. *Examples:*

1. Prove that

$$\tan\frac{A}{2}=\frac{\sin A}{1+\cos A}$$

$$\frac{\sin A}{1+\cos A}=\frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{2\cos^2\frac{A}{2}}\qquad\text{Arts. 21, 22.}$$

$$=\tan\frac{A}{2}$$

It will be observed, that in the above example, the second side of the equation is first operated upon, and that the transformation is much more obvious than if we had commenced with $\tan\frac{A}{2}$. No absolute rule can be given for the treatment of formulæ whose equality has to be proved, but it will generally be found most advantageous to begin with that side of the equation which involves the larger angles.

2. Prove that $\tan\frac{A}{2}=\sqrt{\frac{1-\cos A}{1+\cos A}}$

Is the positive or negative sign of the root to be taken?

Hence find tan 15°, tan 22° 30′, tan 67° 30′, and tan 36°.

3. Prove that $\tan\frac{A}{2}=\frac{1-\cos A}{\sin A}$

4. $\sin 2A=\frac{2\sqrt{\sec^2 A-1}}{\sec^2 A}$

5. $\sin 2A=\cot A-\cot A\,.\cos 2A$

6. $\tan^2\frac{A}{2}=\frac{2\sin A-\sin 2A}{2\sin A+\sin 2A}$

We have

$$\sin 2A=2\sin A\,.\cos A$$

$$\therefore\quad\frac{\sin 2A}{2\sin A}=\cos A$$

$$\therefore\quad\frac{2\sin A-\sin 2A}{2\sin A+\sin 2A}=\frac{1-\cos A}{1+\cos A}$$

$$=\tan^2\frac{A}{2}\text{ by Ex. 2.}$$

7. Prove that

$$\tan(45^\circ+A)=\frac{\cos A+\sin A}{\cos A-\sin A}$$

8. $$\sec 2A=\frac{\sec^2 A}{1-\tan^2 A}$$

9. $$\tan A+\cot A=2\operatorname{cosec} 2A$$

10. $$\tan\left(45^\circ+\frac{A}{2}\right)=\frac{1+\sin A}{\cos A}$$

11. $$\frac{\tan A+\sec A}{\cot A+\operatorname{cosec} A}=\tan\frac{A}{2}\tan\left(45^\circ+\frac{A}{2}\right)$$

(See Examples 1 and 10).

12. Prove that $\left(1+\tan\frac{\alpha}{2}\right)^2=2\,\frac{1+\sin\alpha}{1+\cos\alpha}$

$$\left(1+\tan\frac{\alpha}{2}\right)^2=\sec^2\frac{\alpha}{2}+2\tan\frac{\alpha}{2}$$

whence the result is immediately deducible.

13. Prove that

$$\tan\left(45^\circ-\frac{A}{2}\right)+\cot\left(45^\circ-\frac{A}{2}\right)=2\sec A$$

14. $\tan(\alpha+30^\circ)\tan(\alpha-30^\circ)=\frac{1-2\cos 2\alpha}{1+2\cos 2\alpha}$

$$\frac{1-2\cos 2\alpha}{1+2\cos 2\alpha}=\frac{3\sin^2\alpha-\cos^2\alpha}{3\cos^2\alpha-\sin^2\alpha}$$

$$=\frac{\sqrt{3}\sin\alpha+\cos\alpha}{\sqrt{3}\cos\alpha-\sin\alpha}\cdot\frac{\sqrt{3}\sin\alpha-\cos\alpha}{\sqrt{3}\cos\alpha+\sin\alpha}$$

$$=\tan(\alpha+30^\circ)\tan(\alpha-30^\circ)$$

15. Prove that

$$\sin(\alpha+\beta)\sin(\alpha-\beta)=\sin^2\alpha-\sin^2\beta$$

16. $$\tan(\alpha+\beta)=\frac{\sin^2\alpha-\sin^2\beta}{\sin\alpha\cos\alpha-\sin\beta\cos\beta}$$

17. $$\sin A\cos(B-C)-\sin B\cos(A-C)=$$
$$=\sin(A-B)\cos C$$

18. $$\frac{\operatorname{cosec} 2A}{1+\operatorname{cosec} 2A}=\frac{1+\tan^2 A}{(1+\tan A)^2}$$

We have $$\sin 2A=\frac{2\sin A\cos A}{\sin^2 A+\cos^2 A}$$
$$=\frac{2\tan A}{1+\tan^2 A}$$

19. Prove that

$$\frac{\cos B-\cos A}{\cos B+\cos A}=\tan\frac{A+B}{2}\tan\frac{A-B}{2}$$

(See Article 25.)

20. If $$A+B+C=2S$$
$$\cos 2S+\cos 2(S-A)+\cos 2(S-B)+$$
$$+\cos 2(S-C)=4\cos A\cos B\cos C$$

21. If $$\frac{\cos u-e}{1-e\cos u}=\cos v$$
$$\tan\frac{v}{2}=\sqrt{\frac{1+e}{1-e}}\cdot\tan\frac{u}{2}$$

22. If $\sin n=\sin a\,.\,\sin(n+y)$

$$\tan n=\frac{\sin a\,.\,\sin y}{1-\sin a\cos y}$$

For $\sin n = \sin a\,(\sin n \cos y + \cos n \sin y)$

$\therefore\ \tan n\,(1 - \sin a \cos y) = \sin a \sin y$

23. If $\tan n = \cos a \,.\, \tan y$

$$\tan(y-n) = \frac{\tan^2\frac{\alpha}{2} \,.\, \sin 2y}{1+\tan^2\frac{a}{2}\cos 2y}$$

24. If $\tan\frac{\alpha}{2} = \tan^3\frac{\beta}{2}$ and $\tan\beta = 2\tan\phi$, ϕ is an arithmetic mean between α and β

For $\tan\frac{\alpha+\beta}{2} = \frac{1}{2}\tan\beta = \tan\phi$

25. Prove that

$\sin^3 A = 4\sin A \,.\, \sin(60^\circ + A)\sin(60^\circ - A)$

26. $\frac{\sin 3A + \cos 3A}{\sin 3A - \cos 3A} = \frac{2\sin 2A + 1}{2\sin 2A - 1} \,.\, \tan(45^\circ - A)$

27. $\tan^2 A - \tan^2 B = \frac{\sin(A-B)\sin(A+B)}{\cos^2 A \,.\, \cos^2 B}$

(See Example 15.)

28. $\cos 2A + \cos 2B = 2\cos(A+B)\cos(A-B)$

29. $2(\sin^2 A \sin^2 B + \cos^2 A \cos^2 B) =$

$= 1 + \cos 2A \cos 2B$

30. $\sin(A+B) =$

$= 2\sin A - \sin(A-B) - 4\sin A \sin^2\frac{B}{2}$

31. $\tan B = \frac{\sin B \,.\, \sin(A+B)}{\sin A + \sin B \cos(A+B)}$

32. $\cos(36^\circ + A)\cos(36^\circ - A) +$

$$+\cos(54^\circ + A)\cos(54^\circ - A) = \cos 2A$$

33. ABC are the angles of a triangle in which,

$$\sin^2 A = \sin^2 B + \sin^2 C.$$

Is the triangle right-angled or not?

34. Prove that $1 + \cos(2A - 2B)\cos 2B =$

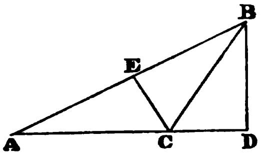
$$= \cos^2 A + \cos^2(A - 2B)$$

35. $$\cot\frac{\alpha}{2} - \tan\frac{\alpha}{2} = 2(\tan\alpha + 2\cot 2\alpha)$$

36. Find the sines, cosines and tangents of the following angles: 75°, 150°, 40°.30′ and 108°.

37. Prove geometrically that

$$\cos 2A = \cos^2 A - \sin^2 A.$$



Let ABC be an isosceles triangle, in which $AC = BC$. Draw CE, BD respectively perpendicular to AB and AC or AC produced.

Then $ABC = BAC = A$ and $BCD = 2A$

By equating two values of sec A we have

$$\frac{AC}{AE} = \frac{AB}{AD}$$

$$\therefore \quad AC\,(AC+CD)=2AE^2$$

$$\therefore \quad AC^2+AC\,.\,CD=2AE^2$$

$$\therefore \quad AC\,.\,CD=2AE^2-AC^2$$

$$=AE^2-EC^2$$

$$\therefore \quad \frac{CD}{BC}=\frac{AE^2}{AC^2}-\frac{EC^2}{AC^2}$$

$$\therefore \quad \cos 2A=\cos^2 A-\sin^2 A$$

If $2A$ is an obtuse angle, the proof is slightly different.

38. Prove geometrically that

$$\sin 2A=2\sin A\cos A$$

39. Prove geometrically that

$$\tan (A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

Construct the figures as in Art. 18. The proof when A, B and $A+B$ are all acute, is as follows:

$$\frac{BD}{AB}=\frac{CE}{AC}$$

$$\therefore \quad \frac{BD}{CD}=\frac{CE\,.\,AB}{CD\,.\,AC}=\frac{CE(BE+AE)}{AC\,.\,CD} \quad \ldots\ldots \quad (1)$$

Again, $$\frac{AD}{AB}=\frac{AE}{AC}$$

$$\therefore \quad AD\,.\,AC=AB\,.\,AE$$

$$\therefore \quad AC^2+AC\,.\,CD=AE^2+AE\,.\,BE$$

$$\therefore \quad AC\,.\,CD=AE\,.\,BE-CE^2.$$

Substituting in equation (1) we have

$$\frac{BD}{CD}=\frac{CE(AE+BE)}{AE\,BE-CE^2}$$

$$=\frac{\frac{CE}{AE}+\frac{CE}{BE}}{1-\frac{CE}{AE}\,\frac{CE}{BE}}$$

$$\therefore\ \tan\,(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

The cases where some of the angles A, B, $A+B$, are obtuse may be treated in a similar manner, with such changes of sign as are required by the use of Figs. II. and III. in Art. 18.

40. Prove geometrically that

$$\tan\,(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$$

Use the figures of Art. 19, and proceed in a manner analogous to the last example.

41. Prove geometrically that

$$\tan 2A=\frac{2\tan A}{1-\tan^2 A}$$

The proof is the same as that in Example 39, but simplified by the equality of AC and BC.

42. Prove that

$$\tan^{-1}\frac{n\cos\phi}{1-n\sin\phi}-\tan^{-1}\frac{n-\sin\phi}{\cos\phi}=\phi$$

For the tangent of the left side of the equation

$$=\frac{\dfrac{n\cos\phi}{1-n\sin\phi}-\dfrac{n-\sin\phi}{\cos\phi}}{1+\dfrac{n\cos\phi(n-\sin\phi)}{(1-n\sin\phi)\cos\phi}}$$

$$=\tan\phi.$$

43. Prove that $\tan^{-1}\frac{e}{\sqrt{1-e^2}}=\sin^{-1}e$

44. $\cos^{-1}\frac{1-m^2}{1+m^2}-\cos^{-1}\frac{1-n^2}{1+n^2}=$

$$=\tan^{-1}\frac{2(m-n)(1+mn)^2}{(1+mn)^2-(m-n)^2}$$

45. If $\sec^{-1}a-\sec^{-1}b=\sec^{-1}\frac{n}{b}-\sec^{-1}\frac{n}{a}$

$$n=\pm a\,.\,b.$$

For $\sec^{-1}a+\sec^{-1}\frac{n}{a}=\sec^{-1}b+\sec^{-1}\frac{n}{b}$

whence by equating the cosines of each side of the equation the result is at once obtained.

46. Shew that the equation $\tan^3 n=\tan(n-a)$ is impossible, unless $\sin a$ is less than $\frac{1}{3}$.

47. If $\sec n=2\sin^2 n+\cos n$

$$n=0^\circ,\ 60^\circ \text{ or } 180^\circ.$$

48. If $\sin 2n=\sin^2 3n$

$$n=15^\circ.$$

49. If $\tan a\,.\tan n=\tan^2(a+n)-\tan^2(a-n)$

$$\cos n=\pm(1\pm\cos a)$$

50. If $\sin^2 2A-\sin^2 A=\sin^2 30^\circ$

$$A=18^\circ.$$

51. The tangents of the angles A, B, C of a triangle are in geometric progression whose ratio is n. Shew that

$$\sin 2C=n\sin 2A.$$

52. If $\operatorname{cosec}^2\dfrac{\phi}{2}-\sec^2\dfrac{\phi}{2}=2\sqrt{3}\operatorname{cosec}^2\phi$

$$\phi=30^\circ.$$

53. If $A+B+C=180^\circ$

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C$$

For $\tan A+\tan B+\tan C=$

$$=\frac{(\sin A\cos B+\sin B\cos A)\cos C+\sin C.\cos A\cos B}{\cos A\cos B\cos C}$$

$$=\frac{\sin C.\cos C+\sin C(\sin A\sin B-\cos C)}{\cos A\cos B\cos C}$$

$$(\text{since } C=180^\circ-\overline{A+B})$$

$$=\tan A\tan B\tan C$$

54. If $\alpha+\beta+\gamma=180^\circ$

$$\frac{\cot\dfrac{\alpha}{2}+\cot\dfrac{\gamma}{2}}{\cot\dfrac{\beta}{2}+\cot\dfrac{\gamma}{2}}=\frac{\sin\beta}{\sin\alpha}$$

55. If $\alpha+\beta+\gamma=180^\circ$

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma=1$$

56. If $\alpha+\beta+\gamma=90^\circ$

$$\frac{\cos\alpha+\sin\gamma-\sin\beta}{\cos\beta+\sin\gamma-\sin\alpha}=\frac{1+\tan\frac{\alpha}{2}}{1+\tan\frac{\beta}{2}}$$

57. If $\tan^2 x+\sec 2x=\dfrac{7\sqrt{3}-10}{\sqrt{3}}$

$$x=60^\circ$$

CHAPTER III.

ON THE NATURE AND USE OF TABLES OF THE LOGARITHMS OF NUMBERS AND TRIGONOMETRICAL FUNCTIONS.

31. Def. If $a^x=N$, x is called the logarithm of the number N to the base a; that is, the logarithm of any number to a given base is the power to which the base must be raised to produce the number. This is commonly expressed by the equation

$$x=\log_a N$$

32. When the logarithms of numbers to any given base are known, multiplication may be performed by means of addition, division by subtraction, involution by multiplication, and evolution by division.

For, let N, N' be two numbers whose logarithms to the base a are x, x'.

$$\therefore \quad N=a^x$$

$$N'=a^{x'}$$

$$\therefore \quad N.N=a^{x+x'}$$

$$\therefore \quad \log_a N . N' = x + x'$$
$$= \log_a N + \log_a N'$$

or the logarithm of the product of two numbers to any base is the sum of the logarithms of the numbers to the same base.

Again $$\frac{N}{N'} = a^{x-x'}$$

$$\therefore \quad \log_a \frac{N}{N'} = x - x'$$
$$= \log_a N - \log_a N'$$

or the logarithm of the quotient of two numbers is the difference of their logarithms.

Again $$N^m = a^{m \cdot x}$$

$$\therefore \quad \log_a N^m = m\, x$$
$$= m \log_a N$$

or the logarithm of a number raised to the power m is m times the logarithm of the number.

Again $$\sqrt[m]{N} = a^{\frac{x}{m}}$$

$$\therefore \quad \log_a \sqrt[m]{N} = \frac{x}{m}$$
$$= \frac{1}{m} \log_a N$$

or the logarithm of the m^{th} root of a number is $\left(\frac{1}{m}\right)^{\text{th}}$ of the logarithm of the number.

If, therefore, we had tables from which we could determine the logarithm of any known number, and the number corresponding to any known logarithm, we could perform the operations of multiplication, division, involution, and evolution, by the shorter processes of addition, subtraction, multiplication, and division respectively.

For this purpose it is immaterial to what base the logarithms are calculated, but the base 10 is the one always employed.

When no base is specified, it will be understood that the base is 10, and $\log_{10}N$ will, for brevity's sake, be written $\log N$.

33. Since $a^1=a$ it follows that

$$\log_a a=1$$

or the logarithm of the base equals unity.

34. Since $a^0=1$

$$\log_a 1=0$$

or the logarithm of unity to any base equals zero.

35. If in the equation

$$a^x=N$$

a is integral and N fractional, x must be negative, since any positive power of a must be greater than unity. Hence the logarithm of a fraction to an integral base must be negative.

36. Since a^x is always positive if a is so, the logarithm of a negative number to a positive base is impossible.

37. *To trace the values of the logarithms of numbers to the base* 10.

$$\begin{array}{rcl} 1 = 10^0 & \therefore & \log 1 = 0 \\ 10 = 10^1 & \therefore & \log 10 = 1 \\ 100 = 10^2 & \therefore & \log 100 = 2 \\ 1000 = 10^3 & \therefore & \log 1000 = 3 \\ \&c. = \&c. & & \&c. = \&c. \end{array}$$

Hence the logarithm of any number between 1 and 10, *i.e.* of any number containing one integral digit only, lies between 0 and 1, that of a number containing two such digits between 1 and 2, and generally that of a number containing n integral digits between $n-1$ and n. Hence a logarithm may generally be expressed by the sum of an integer and a decimal.

The integral part of the number to which any logarithm is equal is called the *characteristic* of the logarithm, and the decimal part is called the *mantissa*. The result above obtained may be expressed by saying that the characteristic of the logarithm of any number is the number of integral digits contained in the given number diminished by unity.

Again, to determine the logarithms of fractional or decimal numbers, we have

$$\begin{array}{lcl} 1 = 10^0 & \therefore & \log 1 = 0 \\ .1 = 10^{-1} & \therefore & \log .1 = -1 \end{array}$$

$$\begin{aligned} .01 &= 10^{-2} \quad \therefore \quad \log .01 = -2 \\ .001 &= 10^{-3} \quad \therefore \quad \log .001 = -3 \\ \&c. &= \&c. \qquad\qquad \&c. = \&c. \end{aligned}$$

Hence the logarithm of a decimal whose first significant digit is in the first decimal place, lies between 0 and -1, that of a decimal having one cipher before the first significant digit, between -1 and -2, and generally that of a decimal having n ciphers before the first significant digit, between $-n$ and $-(n+1)$.

A number between 0 and -1 may obviously be written either as $-d$ or as $-1+d'$ where d, d' are decimals less than unity; and in like manner, a number between $-n$ and $-(n+1)$ may be written either as $-\{n+d\}$ or as $-(n+1)+d'$ where d, d' are, as before, decimals less than unity. The latter is the mode always adopted in writing negative logarithms, as it is convenient to keep the mantissa always positive. Hence the characteristic of the logarithm of a decimal having no ciphers before its first significant digit is -1, and generally that of a decimal having n ciphers before the first significant digit is $-(n+1)$.

By the above rules the characteristic of the logarithm of any positive number, whether greater or less than unity, may be determined by inspection, and the mantissa only need be recorded in the tables.

This is one of the reasons why the base 10 is chosen, as these rules would not hold for any other base.

38. *The mantissæ of the logarithms to the base* 10 *of all numbers having the same significant digits, and differing only in the position of the decimal point, are the same.*

For let N, N' be two numbers having the same significant digits.

Then $N = 10^m \; N'$

or $10^{-m} N'$

where m is a positive integer

$\therefore \; \log N = \log 10^m + \log N'$

or $\log 10^{-m} + \log N'$

$= m + \log N'$

or $-m + \log N'$

That is, the former logarithm may be obtained from the latter by increasing or diminishing the characteristic by m without altering the mantissa. This of course supposes that the division of the logarithm into characteristic and mantissa is so made as to keep the mantissa positive, whether the characteristic be positive or negative, and is the reason why that division is adopted. Hence, when the logarithm of any number is given in the tables, those of all numbers having the same significant digits are known also, and need not be again recorded. This property also is peculiar to the base 10, and is a further reason for employing that base.

39. The following example will illustrate the foregoing observations.

Having given that the mantissa of log 9721 = .98771, to find log 97.21, log 9.721, log .9721 and log .009721.

Since 97.21 contains two integral digits, its characteristic = 1, and its mantissa is the same as that of the given logarithm

$$\therefore \quad \log 97.21 = 1.98771$$

Again, the characteristic of log 9.721 is zero, since it contains one integral digit

$$\therefore \quad \log 9.721 = .98771$$

The characteristic of log .9721 is -1, since there are no ciphers before the first significant digit, and the mantissa is the same as before

$$\therefore \quad \log .9721 = -1 + .98771$$

This is commonly written, for brevity's sake,

$$\log .9721 = \bar{1}.98771$$

the negative sign being placed above the characteristic to shew that it affects the characteristic only.

So also $\quad \log .009721 = \bar{3}.98771$

40. The logarithmic tables more especially referred to in the following articles, are those published by

the Society for the Diffusion of Useful Knowledge.* They give the logarithms (or rather the mantissæ of the logarithms) of all numbers from 1 to 10000, calculated to five decimal places; the logarithm of any number containing four significant digits may, therefore, be taken out of the tables, and conversely if the value of the logarithm is given, the first four significant digits of the number will be given immediately by the tables. Thus, if it is required to find the logarithm of 27.03, we turn to the number 2703 in the tables (which will be found at page 31), and opposite to it, under the head *log*, we find the mantissa of its logarithm recorded as .43185, and as 27.03 contains two integral digits, its characteristic is unity

$$\therefore \quad \log 27.03 = 1.43185$$

Again, suppose we have given

$$\log N = \bar{2}.26631$$

to determine N, we look under the column headed *log* for the mantissa .26631, when we find that the two nearest values given in the tables are .26623 and .26647, opposite to the numbers 1846 and 1847. The number N must therefore lie between these two numbers; that is, its first four significant digits must be 1846. To determine the position of the decimal point, we observe that the characteristic is -2, and therefore that there is one cipher before the first significant digit

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* These tables are published at a very moderate price, and should be in the hands of every student.

$$\therefore \quad N = .01846$$

a result correct to five decimal places.

The question might have occurred in the form, given $\log N = -1.73369$ to find N and in this case we must first reduce the given logarithm to a shape in which its mantissa is positive before referring to the tables.

Thus

$$\begin{aligned} \log N &= -1.73369 \\ &= -1 - .73369 \\ &= -2 + \{1 - .73369\} \\ &= \bar{2}.26631 \end{aligned}$$

whence N may be found as before.

41. If it is required to find the logarithm of a number containing more than four significant digits, or conversely from a given logarithm to determine more than four of the significant digits of the corresponding number, we must proceed as in the following examples.

1. To find log 18464,

log 18464 lies between log 18460 and log 18470. By referring to the tables, we find that the mantissa

of $\log 1846 = .26623$

and of $\log 1847 = .26647$

$$\therefore \quad \log 18460 = 4.26623$$
and
$$\log 18470 = 4.26647$$

The required logarithm obviously lies between these values. By subtracting the former from the latter we find

$$\log 18470 - \log 18460 = .00024$$

that is, an increase of 10 in the number corresponds to an increase of .00024 in the logarithm. If we assume that the increase of the logarithm is proportional to that of the number, it will follow that when the number 18460 is increased by 4 to give the proposed number 18464, the logarithm ought to be increased by $\left(\frac{4}{10}\right)^{\text{ths}}$ of .00024, the increase corresponding to an addition of 10 to the number.

$$\text{Hence} \quad \log 18464 - \log 18460 = \frac{4}{10}(.00024)$$
$$= .000096$$

$$\begin{aligned} \therefore \quad \log 18464 &= 4.26623 \\ &\underline{+ .000096} \\ &= 4.266326 \end{aligned}$$

The last digit of course cannot be depended on, because the corresponding digit in the value of log 18460 is not given, but when it is greater than 5, the next preceding digit should be increased by unity, which will give a nearer approximation to the true logarithm than if its actual value were retained.

Thus, in the above example we should have log 18464=4.26633.

The assumption above made, that the increase of the logarithm is proportional to that of the number is called *the principle of proportional parts.* If it were strictly true, it would follow that successive additions of 10 to the number would produce successive equal additions to the logarithm. Thus, since

$$\log 18470 - \log 18460 = .00024$$

we should have

$$\log 18480 - \log 18460 = 2(.00024)$$

and $\therefore$ $\log 18480 - \log 18470 = .00024$

Now the mantissæ of these latter logarithms are .26647 and .26670, whose difference is .00023, and not .00024. This shews that the assumption is not strictly true, but the nearness of the two differences indicates that it is nearly so, and that the value found upon that assumption for the proposed logarithm is approximately correct. Upon examining the tables, this will be found to be the case throughout. The significant figures of the differences are set down under the column headed *D*, and it will be seen that although they are not constant, as they would be if the principle of proportional parts were strictly true, yet the variations are so gradual that no error of importance can arise from assuming its truth for the purpose of determining logarithms intermediate to those given by the tables.

The rule to be followed in finding the logarithm of a number containing five significant digits will therefore be: *Take out from the tables the mantissa of the logarithm of the first four digits of the number, and add that proportion of the difference placed next below it, which the fifth digit bears to ten. This will give the mantissa of the proposed logarithm. To find the logarithm itself, we must prefix the characteristic as determined by inspection.*

No confusion can arise from the fact that only the significant digits of the difference are given in the column of differences, if it is remembered that the last digit of the difference is always in the fifth decimal place.

The following example is worked in the form which will be found most convenient in practice.

To find log 163.45

Mantissa of log 1634 $= .21325$

Diff. for 10 = 27

Diff. for $5 = \frac{1}{2}(27) =$ 135

∴ mantissa of log 16345 $= .21338$

∴ log 163.45 $= 2.21338$

If there are six digits in the proposed number, we must proceed in the same way, adding the further proportion of the difference which is due to the digit in the sixth place. This will obviously be $\frac{1}{10}$th of that which would correspond to the same digit in the fifth place, and is therefore given by finding this latter difference and placing it one place further to the right.

Thus, to find log 163.454

Mantissa of log 1634 $=.21325$

Diff. for $10=27$

$\therefore$ Diff. for 5 $=\frac{5}{10}(27) =$ 135

04 $=\frac{1}{10}\cdot\frac{4}{10}(27)=$ 108

$\therefore$ Mant. of log 163454 $=.2133958$

$\therefore$ log $163.454=2.21339$

It is useless to add the difference for any digit beyond the sixth, as our results cannot be depended on beyond five places of decimals, and such a difference will never affect the figures in the fifth place.

To find log 3286687.

Mantissa of log 3286 $=.51667$

Diff. for $10=13$

Diff. for 6 $= \frac{6}{10}(13)=$ 78

$08=\frac{1}{10}\frac{8}{10}\cdot(13)=$ 104

$\therefore$ Mantissa of log 328668 $=.5167584$

$\therefore$ log. $3286687=6.51676$

6 being the characteristic corresponding to the seven integral digits in the number.

42. The principle of proportional parts will also enable us to solve the converse problem; namely, to determine the number as far as five or six significant digits, when its logarithm is given, as will be easily understood by the following example.

Given $\log N = 3.76323$ to find N.

By referring to the tables, we find that the mantissa .76323 lies between the given mantissæ .76320 and .76328, corresponding to the numbers 5797 and 5798, and that the significant part of the corresponding difference is 8.

$$\text{N}^0 \text{ whose mant is } .76320 \qquad = 5797$$

$$8 = \text{diff. for } 10 \quad \therefore \quad 3 = \text{diff. for } \frac{3}{8} 10 \text{ or} \qquad 37$$

$$\therefore \text{N}^0 \text{ whose mant is } .76323 \qquad = 579737$$

$$\therefore \quad \text{Number whose logarithm is } 3.76323 = 5797.37$$

The only point as to which the student could have any difficulty in this example, is the position to be assigned to the figures of the difference 37. Since, however, the difference 8 is the difference between the mantissæ of the logarithms of 5797 and 5798, it is clear that the 10, for which 8 is said to be the difference, means 10 so placed that the unit shall fall under the 7 of the previous number;

that is, in the fourth place; and, therefore, when $\left(\frac{3}{8}\right)^{\text{ths}}$ of this is taken, the first figure of the result must fall under the cipher of the 10, or in the fifth place, as in the example.

It is easily seen, from this example, that the rule to be applied in all cases is as follows: *Select from the tables the number corresponding to the mantissa next less than that of the proposed logarithm, and for the fifth and following digits add that proportion of* 10 *which the difference of the mantissæ of the proposed and selected logarithms bears to the difference recorded next below the latter. Point off the number of decimal places indicated by the characteristic of the proposed logarithm.*

Ex. Find the number whose logarithm $=8.27956$

N^0 whose mant is $.27944 \qquad =1903$

$23=$ diff. for $10 \;\therefore\; 12=$ diff. for $\frac{12}{23}10$ or $\qquad 52$

$\therefore N^0$ whose mant is $.27956 \qquad =190352$

$\therefore$ Number whose log is $8.27956=19035200$

the ciphers being added to give 9 integral digits corresponding to the characteristic 8.

43. *Examples:*

1. Given log $73810=4.86812$
log $2.896=.4618$
log $213753=5.32992$

Multiply 73810 by 2.896.

We have

$$\begin{aligned}\log 73810 \times 2.896 &= \log 73810 + \log 2.896\\ &= 4.86812 + .4618\\ &= 5.32992\\ &= \log 213753\end{aligned}$$

$$\therefore\quad 73810 \times 2.896 = 213753$$

2. With the same data divide 213753 by 2.896.

$$\begin{aligned}\log \frac{213753}{2.896} &= \log 213753 - \log 2.896\\ &= 5.32992 - .4618\\ &= 4.86812\\ &= \log 73810\end{aligned}$$

$$\therefore\quad \frac{213753}{2.896} = 73810$$

3. Given $\log 3 = .47712$

$\log 19683 = 4.29408$

Find the value of 3^9.

$$\begin{aligned}\log 3^9 &= 9 \times \log 3\\ &= 9 \times .47712\\ &= 4.29408\\ &= \log 19683\end{aligned}$$

$$\therefore\quad 3^9 = 19683$$

4. Given $\log 2 = .30103$

$\log 1.1892 = .07525$

Find the value of $\sqrt[4]{2}$.

$$\begin{aligned}\log \sqrt[4]{2} &= \frac{1}{4}\log 2\\ &= \frac{1}{4}(.30103) = .07525\\ &= \log 1.1892\end{aligned}$$

$$\therefore \quad \sqrt[4]{2} = 1.1892$$

5. Given $\log 2 = .30103$
$\log 3 = .47712$

Find the logarithms of 5, 25, 75, 540, and 360.

$$\begin{aligned}\log 5 &= \log 10 - \log 2\\ &= 1 - .30103\\ &= .69897\end{aligned}$$

and the logarithms of the remaining numbers may, in like manner, be expressed in terms of the given logarithms.

The following examples must be worked with the assistance of the tables.

6. Multiply together 84.735 and .0852.

$$\begin{array}{lrcr} \text{Mantissa of log} & 84.730 & = & .92804\\ \text{Diff. for } 10 = 5 \;\therefore\; \text{Diff. for} & 5 & = & 25\\ \hline \therefore \quad \log & 84.735 & = & 1.928065\\ \text{Also} \quad \log & .0852 & = & \bar{2}.93044\\ \therefore\; \log 84.735 \times .0852 & & = & .858505 \end{array}$$

Again $.85848 = \log 7.219$

$6 = \text{diff. for } 10 \; \therefore \quad 2 = \text{diff. for } 4$

$\therefore \quad .85850 = \log 7.2194$

$\therefore \quad \log 84.735 \times .0852 = \log 7.2194$

$\therefore \quad 84.735 \times .0852 = 7.2194$

7. Multiply 32.95 by 4.6819
.0175 by .0032
63894 by 729.53
107 by .00175

8. Divide the numbers in the first column by those in the second, respectively.

In the following examples, the results given are those obtained directly, without the aid of logarithms, and are absolutely correct. The values obtained by the use of the tables are liable to small errors, because approximate values only of the logarithms are given, all figures after the fifth decimal place being omitted. The extent of the error will be seen to increase with the number of operations. Thus, in Ex. 11. it reaches the third decimal place.

9. Find the product of the square roots of

1505.28 and 52.92

Also of 752640 and .10584
25.088 and 3175.2
1.0752 and 74088
376.32 and 211.68

The result will be the same in each case, subject only to the errors introduced by using the tables.

10. Extract the square root of 2 and the cube root of 3, and raise their product to the power of 3. The result should equal 6 times $\sqrt{2}$.

11. Extract the cube roots of 4, 10, .2, and .125, and find the product of the results. *Ans.* 1.

12. Divide the square of 84×45 by the square of 1260. *Ans.* 9.

44. In order to apply logarithmic calculations to the formulæ of trigonometry, it is necessary to know not only the logarithms of any numbers that we may require as given by the common logarithmic tables, but also the logarithms of the different trigonometrical functions of any angles. For this purpose, tables of the logarithms to the base 10 of sines, cosines, tangents and cotangents, are constructed, and are used in a manner very similar to the mode already explained of using the common logarithmic tables. Since the sines and cosines of all angles are less than unity, their logarithms will be negative. To avoid the inconvenience of registering negative numbers, all the logarithms recorded in the tables are increased by 10, and called *tabular logarithms.* These tabular logarithms will be denoted by the letter L, as L sin 45°, L tan 15°, &c.,

and the true logarithms will be given by the equations

$$\log \sin A = L \sin A - 10$$
$$\log \cos A = L \cos A - 10$$
$$\log \tan A = L \tan A - 10$$
$$\log \cot A = L \cot A - 10$$

In the tables already mentioned will be found the tabular logarithms of the sines, cosines, tangents and cotangents of all angles, at intervals of one minute, from 0° to 45°. Those of angles between 45° and 90° are really included in the above, because the sine and tangent of an angle greater than 45° are equal respectively to the cosine and cotangent of its complement, which must be an angle less than 45°, and so also the cosine and cotangent of any angle greater than 45° are respectively equal to the sine and tangent of some angle less than 45°.

Thus, for instance, if we require the value of $L \sin 52° 10'$, the complementary angle is 37° 50′, and we look in the tables for $L \cos 37° 50'$, which we find to be 9.89752. To facilitate this method, the degrees and minutes of the complementary angle are written at the bottom and up the last column of the page, opposite to the degrees and minutes of the smaller angle, which are placed at the top and down the first column, and the complementary function is written at the bottom of each column. Thus, in the above example, we look for 52° at the bottom of the page, and then in the column at the bottom of which the word sine is placed, seek for the logarithm opposite to the number 10 in the last column, when we are led to the same number as before.

In this way we may find the logarithms of the sines, cosines, tangents and cotangents of all angles between 0° and 90° containing an exact number of degrees and minutes. The secants and cosecants are not included in the tables, but their logarithms may be immediately found from those of the cosines and sines.

For $$\sec A = \frac{1}{\cos A}$$

$$\begin{aligned} \therefore \quad \log \sec A &= -\log \cos A \\ &= -\{L \cos A - 10\} \\ &= 10 - L \cos A \end{aligned}$$

and in like manner we obtain

$$\log \operatorname{cosec} A = 10 - L \sin A$$

Angles greater than 90° are not included in the tables, because the functions of such angles may always be expressed in terms of those of angles less than 90°, as we have seen in Art. 5.

45. To find the tabular logarithm of an angle which does not consist of an exact number of degrees and minutes, and which is therefore not

exactly given in the tables, we must employ the principle of proportional parts, in the same way as in finding the logarithms of numbers; and the converse problem, viz., having given the tabular logarithm of one of the functions of an angle, to find the number of degrees, minutes, and seconds that it contains, must be solved in the same manner.

Tables of the differences for one minute are placed after the columns containing the tabular logarithms of the sines, cosines, and tangents. The column of differences for cotangents is omitted, because these differences are always the same as for tangents, and are therefore placed between the logarithms of tangents and cotangents, to serve for both.

That the differences for tangents and cotangents are the same, appears as follows. Let A, $A+1'$ be any two consecutive angles.

Then $$\tan A=\frac{1}{\cot A}$$

$$\therefore \quad \log \tan A=-\log \cot A$$

$$\therefore \quad L \tan A=20-L \cot A$$

similarly $$L \tan (A+1')=20-L \cot (A+1')$$

$$\therefore L\tan (A+1')-L\tan A=L\cot A-L\cot (A+1')$$

or the differences between consecutive values of L tan A and L cot A are equal. It will be observed, however, that L cot A decreases, while

L tan *A* increases. So also *L* cos *A* decreases while *L* sin *A* increases.

That the principle of proportional parts is approximately correct, will be seen as before, from the fact that the values of the differences generally vary by slow degrees. In the case, however, of the sine of a very small angle, the cosine of an angle nearly equal to 90°, and the tangent or cotangent of an angle which is either small or nearly equal to 90°, it will be found that the differences vary very rapidly, so that the principle of proportional parts would not give even an approximation to the true value, where the angle is one not exactly given by the tables. To obviate this difficulty, additional tables are calculated of the *L* sines of angles less than 10′, at intervals of 1″, and of angles less than 1° at intervals of .1′ or 6″, so as to dispense with the necessity of applying the principle of proportional parts in the former case, and to reduce the error from this source in the latter. Such a table will be found at the end of the tables above referred to.

46. The following examples will serve to explain the method of determining the tabular logarithms of angles not exactly given in the tables and the converse.

To find log sin 21° 40′ 20″

The required angle lies between the angles 21° 40′ and 21° 41′.

By referring to the tables, we find

$$L \sin 21^\circ\ 40' = 9.56727$$

and from the column of differences we see that the significant figures of the difference for one minute $=32$, the last digit of which corresponds to the fifth place of decimals.

By the principle of proportional parts we have

$$\text{Diff. for } 20'' = \frac{20}{60} \cdot \text{Diff. for } 1' = \frac{1}{3}\,.00032$$
$$= .000106$$

Adding this to the given value of $L \sin 21^\circ\ 40'$, we obtain

$$\begin{aligned} L \sin 21^\circ\ 40'\ 20'' &= \ 9.56727 \\ &\ + .000106 \\ &= \ 9.567376 \end{aligned}$$

$$\begin{aligned} \therefore\quad \log \sin 21^\circ\ 40'\ 20'' &= L \sin 21^\circ\ 40'\ 20'' - 10 \\ &= \bar{1}.567376 \\ \text{or}\qquad & -.432624 \end{aligned}$$

This result (like all others obtained from the tables) can be depended on only as far as five decimal places.

The next example is worked in the form which will be found most convenient. It will be observed that we begin with the angle next greater, instead of that next less than the proposed angle, because the value of L cosine diminishes when the angle increases; and, therefore, if we had started from the lesser angle, the difference would have had to be subtracted instead of added, which is less convenient. This course should always be taken with L cosecant as well as L cosine, as the same reason applies.

To find $L \cos 74^\circ\ 15'\ 50''$.

$L \cos 74^\circ\ 16' \qquad = 9.43323$

Diff. for $1' = 44\ \therefore$ Diff. for $10'' = \frac{1}{6}\,44 = \qquad 73$

$\therefore\ L \cos 74^\circ\ 15'\ 50'' \qquad = 9.433303$

If the true logarithm is required, it may be found by subtracting 10 from the above value.

The following are examples of the converse problem:

Given $L \tan A = 9.6815$ to find A.

In the column headed tangent, the recorded number next less than the proposed, viz.

$9.68142 = L \tan 25^\circ\ 39'$

$32 = \text{diff. for } 1' \quad \therefore \quad 8 = \text{diff. for} \qquad 15''$

$\therefore\ 9.6815 \ = L \tan 25^\circ\ 39'\ 15''$

$\therefore$ the required angle is $25^\circ\ 39'\ 15''$

Given $L \cot A = 9.742$ to find A.

Beginning with the logarithm next *greater* than the proposed, for the reason already given, we have

$9.74226 = \ L \cot 61^\circ\ 5'$

$30 = \text{diff. for } 1' \ \therefore \qquad 26 = \text{diff. for} \qquad 52''$

$\therefore\ 9.742 \ = \ L \cot 61^\circ\ 5'\ 52''$

47. *On the choice of logarithmic formulæ.*

It frequently happens in trigonometrical problems, that a required angle may be determined by using any one of several formulæ. Instances of this will be found in the next chapter.

A glance at the tables will shew that it is often very material to choose the most advantageous formula. Thus, suppose we have, to determine the angle A, the two formulæ

$$L \cos A = 9.99989$$
$$L \sin A = 8.3513$$

If we use the first formula, we find from the tables that A may be any angle between 1° 16′ and 1° 19′, since the first five decimal places in the value of $L \cos A$ are the same for all angles between those limits. This leaves a possibility of an error in the angle chosen of about 2′. If, however, we use the second equation, we find the value of A to be 1° 17′ 12″.

The above is an extreme case, being one where the difference of $L \cos A$ has no sensible digit in the first five places of decimals to which the tables are confined, but an ambiguity of a similar kind must arise in determining the exact number of seconds contained in an angle in all cases where the difference between successive values of the particular function employed is small. If, for example, $L \cos A = 9.99485$, we find that $L \cos 8° 48' =$

9.99486 and L cos 8° 49′ = 9.99484, and the L cosines of all angles between these limits must be represented by one of the numbers 9.99486, 9.99485, or 9.99484. Each of these must correspond to a series of angles, differing by many seconds, and the value of A determined from the above equation is therefore subject to a proportionate error. The general rule for obviating this uncertainty is to determine angles wherever it is possible, by those functions whose differences are the largest. It will be seen by referring to the tables, that for angles less than 45° the difference of the L sine is greater than that of the L cosine, while for angles greater than 45° the reverse is the case, and that the difference of the L tangent and L cotangent is never inconveniently small.

So far, therefore, as the objection in question is concerned, formulæ involving tangents and cotangents may be employed for the determination of angles of any magnitude; formulæ involving the sines or cosines should be preferred, according as the angle is less or greater than 45°. Attention to this becomes of course the more important as the difference of the angle from 45° increases.

There is another consideration which should influence the choice of formulæ, viz., that in certain cases the variations in the differences are so great, that angles intermediate to those exactly given in the tables cannot be accurately determined by the principle of proportional parts. This occurs, as appears by referring to the tables, when a very small angle is determined by its sine, tangent, or cotangent, or an angle nearly 90° by its cosine, tangent, or cotangent. In the case of the sine, the difficulty is obviated, as before stated, by the construction of additional tables of L sine at smaller intervals for angles less than 1°, which of course are also tables of L cosine for angles between 89° and 90°. Such tables are not formed for tangents and cotangents, the use of which must therefore be avoided when the angle is very small or very nearly equal to 90°. From the foregoing remarks we deduce the following rules.

1. *Very small angles must be determined by the sine, the additional table of L sines being used, and angles very nearly equal to* 90° *by the cosine.*

2. *The sine or cosine should be preferred according as the angle is less or greater than* 45°.

3. *The tangent or cotangent may be used for all angles sufficiently remote from* 0° *or* 90° *to make the variations in their successive differences small.*

48. When the formulæ for the determination of angles present themselves under a disadvantageous form, they may be modified as follows:

1. Suppose we have $\cos A = a$

where a is a number nearly equal to unity, and therefore A a small angle.

We have $$\sin\frac{A}{2}=\sqrt{\frac{1}{2}(1-\cos A)}$$

$$=\sqrt{\frac{1-a}{2}}$$

$$\therefore\ L\sin\frac{A}{2}=\frac{1}{2}\log(1-a)-\frac{1}{2}\log 2+10$$

a form in which the angle is determined by the sine of a small angle, and which may therefore be advantageously employed.

2. Suppose we have $\sin A=a$, where a is nearly equal to unity, and therefore A nearly equal to 90°.

This is identical with

$$\cos(90^\circ-A)=a$$

from which we obtain as before,

$$L\sin\frac{90^\circ-A}{2}=\frac{1}{2}\log(1-a)-\frac{1}{2}\log 2+10$$

where $\frac{90^\circ-A}{2}$ is evidently a small angle.

3. Suppose we have $\tan A=a$, where a and therefore A is very small.

We have $$\tan(45^\circ-A)=\frac{1-\tan A}{1+\tan A}$$

$$=\frac{1-a}{1+a}$$

$$\therefore\ L\tan(45^\circ-A)=\log(1-a)-\log(1+a)+10$$

where $(45^\circ-A)$ being nearly equal to 45°, the formula may be advantageously used.

So also if $\tan A=a$, where a is very large, and therefore A nearly equal to 90°.

$$\tan(A-45^\circ)=\frac{\tan A-1}{\tan A+1}$$

$$=\frac{a-1}{a+1}$$

$$\therefore\ L\tan(A-45^\circ)=\log(a-1)-\log(a+1)+10$$

4. Suppose we have $\cot A=a$ where a is small, and therefore A nearly equal to 90°.

$$\tan(A-45^\circ)=\frac{\tan A-1}{\tan A+1}=\frac{1-\cot A}{1+\cot A}$$

$$=\frac{1-a}{1+a}$$

$$\therefore\ L\tan(A-45^\circ)=\log(1-a)-\log(1+a)+10$$

and if $\cot A=a$ where a is very large and therefore A very small

$$\tan(45^\circ-A)=\frac{1-\tan A}{1+\tan A}=\frac{\cot A-1}{\cot A+1}$$

$$=\frac{a-1}{a+1}$$

$$\therefore\ L\tan(45^\circ-A)=\log(a-1)-\log(a+1)+10$$

which formulæ are free from any inconvenience.

49. *On the use of subsidiary angles.*

The only formulæ to which logarithmic calculation is applicable, are those which involve no other arithmetical operations than multiplication, division, involution and evolution. Formulæ which require addition or subtraction must be modified before we can apply logarithmic tables to their calculation. This may sometimes be done by algebraical transformations, of which an instance will be seen in Art. 57, 1. In other cases the object is effected by the introduction of new angles, called subsidiary angles. Suppose, for example, it is required to adapt to logarithmic calculation the equation

$c = a \sin A \pm b \cos A$, a and b being positive.

We have

$$c = a\left\{\sin A \pm \frac{b}{a}\cos A\right\}$$

Now whatever may be the value of $\frac{b}{a}$ there must be some angle whose tangent is equal to it. Let, therefore,

$$\tan \phi = \frac{b}{a}$$

$$\therefore\ L \tan \phi = \log b - \log a + 10 \;.\;.\;.\;.\;.\;.\; (1)$$

and substituting for $\frac{b}{a}$ in the value of c

$$c=a\{\sin A\pm\tan\phi\cos A\}$$

$$=a\sec\phi\sin(A\pm\phi)$$

$$\therefore\ \log c=\log a+L\sec\phi+L\sin(A\pm\phi)-20\quad.\quad(2)$$

From equation (1) ϕ may be found, and then c is known from (2).

Another instance of the use of a subsidiary angle will be found in Art. 57, 2.

50. *Examples :*

1. Given $\sin A=\dfrac{2.193}{1.984}\sin 30^\circ\,10'\,20''$ to find A

We have

$$L\sin A=\log 2.193-\log 1.984+L\sin 30^\circ\,10'\,20''$$

$$L\sin 30^\circ\,10'=9.70115$$

$$22=\text{diff. for }1'\ \therefore\ \text{Diff. for }20''=\quad 73$$

$$\therefore\ L\sin 30^\circ\,10'\,20''=9.701223$$

Also

$$\log 2.193=\ .34104$$

$$\log 1.984=\ .29754$$

$$\therefore\ L\sin A=9.701223+.34104-.29754$$

$$=9.744723$$

$$9.74455=L\sin\ 33^\circ\,44'$$

$$19=\text{diff. for }1'\ \therefore\ 17=\text{diff. for}\quad 54'$$

$$\therefore\ 9.74472=L\sin\ 33^\circ\,44'\,54''$$

$$\therefore\ A=33^\circ\,44'\,54''$$

2. Given $\sin A = \frac{3}{4}\cos 60^\circ\, 30'\, 6''$ to find A.

$$A = 21^\circ\, 40'\, 28''$$

3. Given $\tan A = \frac{2}{2011}\cot 35^\circ$ to find A.

$$\begin{aligned} \therefore\quad L\tan A &= \log 2 - \log 2011 + L\cot 35^\circ \\ &= .30103 - 3.30341 + 10.15477 \\ &= 7.15239 \end{aligned}$$

On turning to the tables, it will be found that the angle lies between 4′ and 5′, but that the successive differences are so far from equal, that the principle of proportional parts cannot be relied upon to give the number of seconds. The equation must therefore be modified as in Art. 48, for which purpose we must first find the numerical value of tan A.

We have $\log \tan A = \bar{3}.15239$

And $.15229 = \text{mantissa of } \log 1420$

$30 = \text{diff. for } 10 \;\therefore\; 10 = \text{diff. for } 3$

$$\therefore\quad .15239 = \text{mantissa of } \log 14203$$

$$\therefore\quad \tan A = .0014203$$

$$\therefore\quad \tan(45^\circ - A) = \frac{1 - .0014203}{1 + .0014203}$$

$$\begin{aligned} \therefore\quad L\tan.(45^\circ - A) &= \log .99858 - \log 1.00142 + 10 \\ &= \bar{1}.99938 - .00061 + 10 \\ &= 9.99877 \end{aligned}$$

Whence we obtain

$$45^\circ - A = 44^\circ\ 55'\ 7''$$

$$\therefore\quad A = \qquad 4'\ 53''$$

4. Adapt to logarithmic calculation the equation

$$r = \sqrt{a^2 \cos^2 A - b^2 \sin^2 B}$$

The most convenient assumption is

$$\frac{b^2}{a^2} \cdot \frac{\sin^2 B}{\cos^2 A} = \sin^2 \theta$$

which is always possible if r is so.

5. Adapt to logarithmic calculation the equations

$$\sin L = \cos A \cos B + \sin A \sin B \cos C$$

and $\qquad x^2 = a^2 \sin^2 A - b^2 \cos^2 A.$

CHAPTER IV.

ON THE SOLUTION OF TRIANGLES.

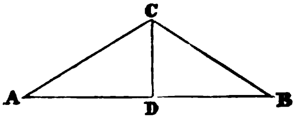
51. A **TRIANGLE** consists of six parts, three sides and three angles, any three of which (except the three angles) being given, the others may be determined. That the remaining parts of a triangle, of which three are given, can generally have but one set of values, is proved in several cases, by the 4th, 7th, 8th, and 26th propositions of the first book of Euclid, and will be generally proved in Art. 53.

The methods of this chapter, however, will not only shew that such determinate values exist, but will enable us to calculate them numerically. Thus the 4th proposition of the first book of Euclid proves that if two sides of a triangle and the included angle have fixed values, the base and remaining angles can have but one determinate set of values. In Art. 56 of this chapter, formulæ are investigated for calculating these values numerically.

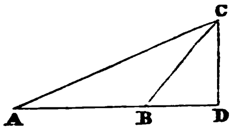
The angles of a triangle ABC are generally denoted by the letters A, B, C, and the sides respectively opposite to them by the letters a, b, c.

52. *The sides of any triangle are proportional to the sines of the opposite angles.*

I.



II.



Let ABC be the triangle; from C draw CD perpendicular to AB or AB produced. In the first figure, A and B are both acute; in the second, one is acute and the other obtuse. Then with either figure

$$\sin A = \frac{CD}{b}$$

$$\sin B = \frac{CD}{a}$$

$$\therefore \quad \frac{\sin A}{\sin B} = \frac{a}{b} \quad \ldots\ldots\ldots \quad (1)$$

In like manner we may obtain

$$\frac{\sin A}{\sin C} = \frac{a}{c} \quad \ldots\ldots\ldots \quad (2)$$

and

$$\frac{\sin B}{\sin C} = \frac{b}{c} \quad \ldots\ldots\ldots \quad (3)$$

this last equation being included in the two former and deducible from them.

53. From Euclid, Book I. Prop. 32, we have also the equation

$$A + B + C = 180^\circ$$

which together with any two of the equations found in the last article, gives us three equations among the six parts A, B, C, a, b, c of the triangle. If, therefore, three of these parts are given, the remaining three may generally be determined.

If the three given parts are the three angles, the last of our equations becomes identical, and we have only two independent equations to determine a, b, and c. In this case, therefore, an indefinite number of triangles having the three given angles, may be constructed, which will be similar, since equations 1 and 2 fix the ratio of the sides. This appears also from Euc. VI. 4.

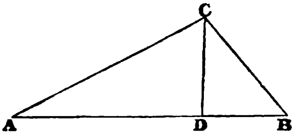
The above equations would suffice in every other case for the solution of the triangle, that is, the determination of its unknown parts; but the formulæ of the next article are in certain cases more convenient in practice.

54. *To express the cosine of an angle of a triangle in terms of its sides.*

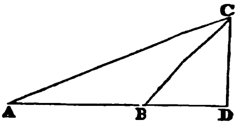
Let A be the angle whose cosine is required.

Draw CD perpendicular to AB or AB produced.

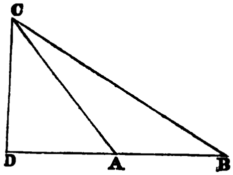
I.



II.



III.



Then with Figs. I. and II. where A is acute

$BC^2 = AB^2 + AC^2 - 2\,AB\,.\,AD$ Euc. II. 13.

and with Fig. III.

$BC^2 = AB^2 + AC^2 + 2AB\,.\,AD$ Euc. II. 12.

and in Figs. I. and II. $\dfrac{AD}{AC} = \cos A$

and in Fig. III. $\dfrac{AD}{AC} = \cos CAD = -\cos A$

In each case therefore we have

$$BC^2 = AB^2 + AC^2 - 2\,AB\,.\,AC \cos A$$

or $$a^2 = b^2 + c^2 - 2\,bc \cos A$$

$$\therefore \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

In like manner $$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

55. The distinct cases which occur in the solution of triangles will now be separately considered. The most simple class of cases is that where the triangle is right angled, in which if any two parts besides the right angle are given, the triangle may be solved. This gives for the solution of right-angled triangles five distinct cases:

1. Where the given parts are two sides.
2. Where they are one side and the hypothenuse.
3. Where they are a side and the opposite angle.
4. Where they are a side and the adjacent angle.
5. Where they are the hypothenuse and an angle.

It will be understood that in speaking of right-angled triangles, the term *side* is applied exclusively to the sides adjacent to the right angle.

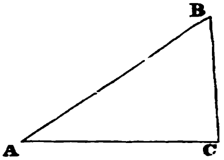
The distinct cases in the solution of oblique-angled triangles are the five following:

1. Where the given parts are the three sides.
2. Where they are two sides and the included angle.

3. Where they are two sides and an angle opposite to one of them.
4. Where they are one side and the two adjacent angles.
5. Where they are one side and one adjacent angle and the opposite angle.

56. *On the solution of right-angled triangles.*

I. Let the two sides a, b be given to determine the remaining parts, namely the angles A and B and the hypothenuse c.



We have $\tan A = \frac{a}{b}$

$$\therefore \quad L \tan A = \log a - \log b + 10$$

whence A and $\therefore$ B are known, since $B = 90° - A$.

Again $\frac{a}{c} = \sin A \quad \therefore \quad c = \frac{a}{\sin A}$

$$\therefore \quad \log c = \log a - L \sin A + 10$$ which gives c.

II. Let the side a and the hypothenuse c be given to determine the angles A and B, and the side b.

We have $\sin A = \frac{a}{c}$

$$\therefore \quad L \sin A = \log a - \log c + 10$$

which determines A, and then B is known from the equation $B=90^\circ-A$

Again $$\frac{b}{c}=\cos A$$

$$\therefore \quad \log b=\log c+L \cos A-10$$

which determines b.

III. Let the side a and the opposite angle A be given to determine B, b, and c.

We have $B=90^\circ-A$ which gives B.

Again $$\frac{b}{a}=\cot A$$

$$\therefore \quad \log b=\log a+L \cot A-10$$

which gives b. Also

$$\frac{a}{c}=\sin A$$

$$\therefore \quad c=\frac{a}{\sin A}$$

$$\therefore \quad \log c=\log a-L \sin A+10$$

which determines c.

IV. Let the side a and the adjacent angle B be given to determine A, b and c.

We have $A=90^\circ-B$ which gives A.

Again $$\frac{b}{a}=\tan B$$

$$\therefore \quad \log b = \log a + L \tan B - 10$$

which gives b. Again

$$\frac{a}{c} = \cos B$$

$$\therefore \quad c = \frac{a}{\cos B}$$

$$\therefore \quad \log c = \log a - L \cos B + 10$$

which gives c.

V. Let the hypothenuse c and the angle A be given to determine B, a, and b.

B is given by the equation $B = 90^\circ - A$

Also $$\frac{a}{c} = \sin A$$

$$\therefore \quad \log a = \log c + L \sin A - 10$$

which gives a. To determine b we have

$$\frac{b}{c} = \cos A$$

$$\therefore \quad \log b = \log c + L \cos A - 10$$

57. *On the solution of oblique-angled triangles.*

I. Let the three sides a, b, c be given to determine the angles A, B, C.

From Art. 54 we have

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

from which and the corresponding equations for the other angles, A, B, and C might be determined.

The labour of using this formula will be found very great, as it does not admit of the application of logarithms, and we shall proceed to put it in a more convenient shape.

$$2\sin^2\frac{A}{2}=1-\cos A$$

$$=1-\frac{b^2+c^2-a^2}{2bc}$$

$$=\frac{a^2-b^2-c^2+2bc}{2bc}$$

$$\therefore\quad \sin^2\frac{A}{2}=\frac{a^2-(b-c)^2}{4bc}$$

$$=\frac{(a-b+c)\ (a+b-c)}{4bc}$$

If we denote the periphery $a+b+c$ by $2s$, this equation becomes

$$\sin^2\frac{A}{2}=\frac{(s-b)\ (s-c)}{bc}$$

$$\therefore\quad \sin\frac{A}{2}=\sqrt{\frac{\overline{s-b}\,.\,\overline{s-c}}{b\,.\,c}}$$

the positive sign of the root being taken, because $\sin\frac{A}{2}$ cannot be negative.

$$\therefore\quad L\sin\frac{A}{2}=10+\frac{1}{2}\log(s-b)+\frac{1}{2}\log(s-c)-$$

$$-\frac{1}{2}\log b-\frac{1}{2}\log c\ .\ .\ .\ .\ (1)$$

whence A may be determined, since $s=\frac{a+b+c}{2}$ and is known.

Another formula for the determination of A may be obtained as follows:

$$2\cos^2\frac{A}{2}=1+\cos A$$

$$=1+\frac{b^2+c^2-a^2}{2bc}$$

$$\therefore\quad \cos^2\frac{A}{2}=\frac{(b+c)^2-a^2}{4bc}$$

$$=\frac{(b+c+a)\ (b+c-a)}{4bc}$$

or with the same notation as before

$$\cos\frac{A}{2}=\sqrt{\frac{s\,.\,\overline{s-a}}{b\,.\,c}}$$

the positive sign of the root being taken, because $\frac{A}{2}$ is less than 90°, and therefore $\cos\frac{A}{2}$ positive.

$$\therefore\quad L\cos\frac{A}{2}=10+\frac{1}{2}\log s+\frac{1}{2}\log(s-a)-\frac{1}{2}\log b-$$

$$-\frac{1}{2}\log c\ .\ .\ .\ (2)$$

A third form may be obtained as follows:

$$\tan\frac{A}{2}=\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}=\sqrt{\frac{s-b\,.\,s-a}{s\,.\,s-a}}$$

whence $L\tan\frac{A}{2}=\frac{1}{2}\log(s-b)+\frac{1}{2}\log(s-c)-$

$$-\frac{1}{2}\log s-\frac{1}{2}\log(s-a)+10\quad.\;.\;.\quad(3)$$

Again we have

$$\sin A=2\sin\frac{A}{2}\cos\frac{A}{2}$$

$$=2\sqrt{\frac{s-b\,.\,s-c}{b\,.\,c}}\sqrt{\frac{s\,.\,s-a}{b\,.\,c}}$$

$$=\frac{2}{b\,.\,c}\sqrt{s\,.\,s-a\,.\,s-b\,.\,s-c}$$

whence $L\sin A$ may be found; but as the expression would involve six logarithms, while those above found contain only four, the others are preferable.

The above formulæ may be made applicable to the other angles B and C by changing the letters throughout.

If we employ equation (1) and determine $\frac{A}{2}$ from its sine, there will be two values of $\frac{A}{2}$ corresponding to the given value of $L\sin\frac{A}{2}$, one being the angle less than 90° which will be given by the tables, and the other an angle greater than 90° complementary to the former. Since, however, A must be less than 180°, $\frac{A}{2}$ must be less than 90°, and the angle given by the tables is therefore the one sought.

In determining the value of the angle opposite to the largest of the three given sides, equation (2) should be used, since the value of A may approach 180°, and cannot be small, and therefore that of $\frac{A}{2}$ may be nearly equal to 90° and cannot be small.

The angles opposite to the other sides should be determined by equation (1), or the corresponding equation, if B or C is the required angle, as the values of the semi-angles must lie between 0° and 45°.

When any two of the angles have been determined by these formulæ, the third angle is most readily obtained from the equation

$$A+B+C=180°.$$

Equation (3) may be used whenever the angle is not very near to 0° or 90°.

II. Let the sides a and b and the included angle C be given to determine B, C, and c.

We have $$\frac{\sin A}{\sin B}=\frac{a}{b}$$

$$*\quad\therefore\quad \frac{\sin A-\sin B}{\sin A+\sin B}=\frac{a-b}{a+b}$$

$$\therefore\quad \frac{\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}=\frac{a-b}{a+b}$$

$$\therefore\quad \tan\frac{1}{2}(A-B)=\frac{a-b}{a+b}\cdot\tan\frac{1}{2}(A+B)$$

Also $$A+B=180^\circ-C\;.\;.\;.\;.\;.\;(1)$$

$$\therefore\quad \tan\frac{1}{2}(A+B)=\cot\frac{C}{2}$$

$$\therefore\quad \tan\frac{1}{2}(A-B)=\frac{a-b}{a+b}\cot\frac{C}{2}$$

$$\therefore\; L\tan\frac{1}{2}(A-B)=\log(a-b)-\log(a+b)+L\cot\frac{C}{2}$$

From this equation $\frac{1}{2}(A-B)$ may be found, and $A+B$ being given by (1), the two angles A and B are determined.

To determine c we have

$$\frac{c}{a}=\frac{\sin C}{\sin A}$$

$$\therefore\quad \log c=\log a+L\sin C-L\sin A$$

---

* If b is greater than a and $\therefore$ B greater than A, $B-A$ should be determined instead of $A-B$, the first step being then in the form $\frac{\sin B-\sin A}{\sin B+\sin A}=\frac{b-a}{b+a}$

from which c may be found after A has been determined as above.

It is sometimes desirable to determine c without previously finding the values of A and B. This may be done from the equation

$$c^2 = a^2 + b^2 - 2ab \cos C$$

To reduce this equation to a form adapted for logarithmic calculation, we proceed as follows:

$$\begin{aligned} c^2 &= (a-b)^2 + 2ab\,(1 - \cos C) \\ &= (a-b)^2 + 4ab \sin^2 \frac{C}{2} \\ &= (a-b)^2 \left\{1 + \frac{4ab}{(a-b)^2} \cdot \sin^2 \frac{C}{2}\right\} \\ &= (a-b)^2 \left\{1 + \left(\frac{2\sqrt{ab}}{a-b} \cdot \sin \frac{C}{2}\right)^2\right\} \end{aligned}$$

Now whatever be the value of $\frac{2\sqrt{ab}}{a-b} \sin \frac{C}{2}$ there must be some angle whose tangent is exactly equal to it. Let this unknown angle be called ϕ, so that

$$\tan \phi = \frac{2\sqrt{ab}}{a-b} \sin \frac{C}{2}$$

and $\therefore L \tan \phi = \log 2 + \frac{1}{2} \log a + \frac{1}{2} \log b - \log (a-b)$

$$+ L \sin \frac{C}{2}$$

from which equation the value of ϕ may be found.

Substituting in the value of c^2 we have

$$c^2=(a-b)^2\{1+\tan^2\phi\}$$
$$=(a-b)^2\sec^2\phi$$
$$\therefore\ 2\log c=2\log(a-b)+2\{L\sec\phi-10\}$$
$$\therefore\quad \log c=\log(a-b)+L\sec\phi-10$$

which determines the value of c.

III. Let the two sides a b and the angle A opposite to one of them be given, to determine the angles B, C and the side c.

$$\frac{\sin B}{\sin A}=\frac{b}{a}$$
$$\therefore\ L\sin B=\log b-\log a+L\sin A$$

which determines B, and then C is known from the equation

$$C=180^\circ-A-B$$

Again $$\frac{c}{a}=\frac{\sin C}{\sin A}$$
$$\therefore\quad \log c=\log a+L\sin C-L\sin A$$

which determines c after C has been found as above.

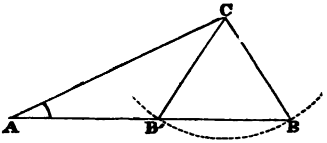
Since by the first formula B is determined by its sine, there will be two values of B corresponding to the value of $L\sin B$ given by the formula, one acute and the other obtuse and supplementary to the former. In the following cases, however, we can discriminate between the two values.

1. If the given angle *A* is obtuse, *B* must be acute, since there cannot be two obtuse angles in a triangle, and the acute value of *B* is therefore the correct one.

2. If the given angle *A* is acute and the side *b* less than the side *a*, it follows that the angle *B* which is opposite to the smaller side, must be less than *A*, Euc. I. 18. In this case, therefore, the acute value of *B* must be taken.

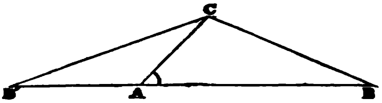
But in the remaining case, viz. where *A* is acute and *b* greater than *a*, we have no means of determining the choice between the two values of *B*.

In fact, in this case both values are correct, as there are then two triangles, which have the given values of *A*, *a* and *b*, in one of which *B* has the acute and in the other the obtuse value given by the formula. This is readily seen by the following geometrical construction.

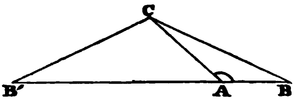


1. Let *A* be acute. Make the angle *BAC* equal to the given angle *A* and *AC* equal to the given side *b*. With center *C* and radius equal to the given side *a*, describe a circle. Then if *a* be less than *b*, this circle will cut *AB* in two points *B*, *B'*, both of which are on the same side of *A*. Either of the two triangles *ABC* or *AB'C* satisfies the given conditions, since each of them contains the given angle

A and the two given sides *a* and *b*; while in the one the angle *B* is acute and in the other *B′* is obtuse and evidently supplementary to *B*. If *a* had been greater than *b*, there would only have been one triangle containing the given elements, since *B′* would then have fallen to the left of *A*, and the triangle *AB′C* would contain an angle supplementary to *A* instead of the given value of that angle, as appears by the figure. In this case, which is the one secondly considered, there is no ambiguity, and the triangle *ABC* with an acute value of *B*, is the only one which satisfies the data of the problem, the same result at which we before arrived.



So also if *A* is obtuse, it is evident that with the same construction as before, *B* and *B′* fall on opposite sides of *a*, and the triangle *ABC* is the only one which can be constructed with the given data, *B* having, as we have already found, an acute value.



On account of the ambiguity above explained the case where two sides and an angle opposite to one of them are given, is called the ambiguous case, but we have seen that these data are sufficient for the determination of the triangle, except when the angle is acute and the side opposite to the given angle less than the other given side.

IV. Let the side a and the adjacent angles B and C be given to determine A, b, and c.

We have $A=180°-B-C$

which determines A.

$$\frac{b}{a}=\frac{\sin B}{\sin A}$$

$$\therefore \quad \log b=\log a+L\sin B-L\sin A$$

which determines b.

And $$\frac{c}{a}=\frac{\sin C}{\sin A}$$

$$\therefore \quad \log c=\log a+L\sin C-L\sin A$$

which determines c.

V. Let the side a, the adjacent angle B and the opposite angle A be given to determine C, b, and c.

We have $C=180°-A-B$

which determines C, and the sides may then be found exactly as in the last case.

58. *Examples:*

1. The sides of a right-angled triangle are 10 feet and 302 ft. 6 in. to find the hypothenuse.

Using the method of Art. 56, I.

$$\begin{aligned}
\text{let } a &= 100 \\
b &= 302.5 \\
\therefore\quad L\tan A &= \log 100 - \log 302.5 + 10 \\
&= 2 - 2.48073 + 10 \\
&= 9.51927 \\
&= 18^\circ\ 17'\ 33'' \\
{}^{*}\ \therefore\quad L\sin A &= 9.49675 \qquad \text{nearly} \\
\therefore\quad \log c &= \log a - L\sin A + 10 \\
&= 2 - 9.49675 + 10 \\
&= 2.50325 \\
\therefore\quad c &= 318.6 \\
&= 318\text{ ft. } 7\text{ in.}
\end{aligned}$$

2. One side of a right-angled triangle $=350$ feet and the hypothenuse $=524$ feet, to determine the angles and the remaining side.

Let $a=350$

$c=524$

Then $A=41^\circ\ 54'\ 30''$

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* The value of $L\sin A$ may be found from $L\tan A$ without first finding the value of A, whereby the labour is slightly diminished when the actual value of A is not required. Thus the nearest value of $L\tan A$ in the tables is 9.51903, and the L sine of the same angle is 9.49654. Also the differences for L sine and L tangent are as 38 to 43, and since the value of $L\tan A$ exceeds that taken from the tables by 24, that of $L\sin A$ must exceed 9.4654 by $\frac{38}{43}\cdot 24=21$, or $L\sin A=9.49675$.

$B = 48° \; 5' \; 30''$

$b = 390.06$ feet.

3. One side of a right-angled triangle $= 21$ ft. 7 in. and the opposite angle $= 30° \; 9' \; 12''$, to determine the remaining parts.

Let $a = 21$ ft. 7 in.

$A = 30° \; 9' \; 12''$

Then $B = 59° \; 50' \; 48''$

$b = 37$ ft. 1 in.

$c = 43$ ft.

4. In a right-angled triangle the side $a = 1000$ feet, and the angle $B = 9° \; 24' \; 20''$, to determine the remaining side and the hypothenuse.

$b = 165$ ft. 7 in.

$c = 1013$ ft. 6 in.

5. In a right-angled triangle the hypothenuse $c = 327$ ft. 4 in. and the angle $B = 41° \; 13'$, to determine the sides.

$b = 215$ ft. 7 in.

$a = 246$ ft. 1 in.

6. The three sides of a triangle are respectively equal to 100 feet, 420 feet and 335 ft. 2 in. Determine the angles.

Let $a = 100$

$b = 420$

$c = 335.17$ nearly.

Since the angles A and C are evidently small and B an obtuse angle, as will easily be seen by roughly constructing the triangle, A and C will be most accurately determined from $\sin\frac{A}{2}$ and $\sin\frac{C}{2}$ and B from $\cos\frac{B}{2}$. Two angles only should be thus determined, and the third found by subtracting the sum of the others from 180°. We find thus:

$$A = 8^\circ\ 5'\ 28''$$
$$B = 143^\circ\ 45'\ 54''$$
$$C = 28^\circ\ 8'\ 38''$$

7. Two sides of a triangle are respectively equal to 324 ft. 3 in. and 571 ft. 6 in. and the included angle to 37° 51′ 30″. Determine the remaining angles and side.

Let $a = 324.25$

$b = 571.5$

$C = 37^\circ\ 51'\ 30''$

Then $A = 32^\circ\ 14'\ 30''$

$B = 109^\circ\ 54'$

$c = 372$ ft. 11 in.

8. Given $A = 15^\circ\ 14'\ 25''$

$b = 10$ feet

$a = 17$ feet

To determine the value of c. Is there any ambiguity in this case?

$c = 26$ ft. 5 in.

9 Given $a = 300$ feet

$B = 86° 23' 13''$

$C = 81° 42' 20''$

Determine b and c.

$b = 1451$ feet

$c = 1438$ ft. 8 in.

10. Given $a = 1000$ feet

$A = 20° 12' 41''$

$B = 73° 10' 12''$

Determine b and c.

$b = 2770$ ft. 6 in.

$c = 2889$ ft. 5 in.

11. Two sides of a triangle are each equal to 524 feet, and the included angle is $43° 27' 6''$. Determine the remaining side.

This may be done as in Ex. 7, but the labour may be much diminished by the consideration that the triangle may be divided into two right-angled triangles, each of which has an angle $= 21° 43' 33''$, the hypothenuse $= 524$ feet, and the side opposite to the known angle equal to half the required side of the original triangle, which will be found to be $= 387$ ft. 11 in.

CHAPTER V.

ON SURVEYING.

59. The principal instruments used in surveying, are the chain, the level, the theodolite, and the sextant.

60. *The Chain* is merely a chain whose length is 22 yards, and is used for measuring distances. It is divided into 100 links, the length of each of which is therefore 7.92 inches.

61. *The Level* is used for determining whether two points are in the same horizontal plane. It is constructed as follows. A tube of glass, the upper surface of which is slightly convex, is nearly filled with spirit, so as to have in it a small bubble of air, and hermetically closed. When the tube is placed horizontally, the bubble stands in the middle of the tube, and when the tube is disturbed from a horizontal position, it moves towards the higher end. When the level is used as a separate instrument, it has a telescope attached to it, so adjusted, that when the bubble of the level is in the middle of the tube, the telescope points in a horizontal direction,

and an object seen in the center of the field of view is therefore in the same horizontal plane with the observer's eye. If it is required to determine the difference of elevation of two neighbouring points, a graduated rod is set up at one and examined through the telescope of the level from the other. The point of the rod which is seen in the center of the field of view when the bubble of the level is in the middle of the tube, is in the same horizontal plane with the telescope, and the difference of altitude of the two points is then known by reading off the graduation of the rod which is seen through the telescope.

62. *The Theodolite* is an instrument used for measuring the angle subtended at the point of observation by two visible points, lying either in the same horizontal or in the same vertical plane with the point of observation. It consists of a telescope capable of rotating both in a horizontal and in a vertical plane, in the former of which motions it carries with it a horizontal and in the latter a vertical circle with graduated rims. The number of degrees, minutes, and seconds through which the telescope and its accompanying circles have revolved is known by means of fixed pointers placed close to the edge of each of the graduated circles. The horizontality of the one circle is ascertained by means of spirit-levels attached to the instrument, by which the verticality of the other is also secured. When the telescope is pointed first

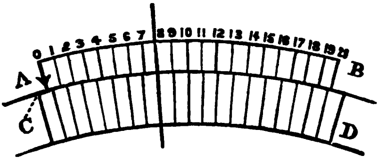
to one object and then to another, in the same horizontal plane, the difference of the readings of the horizontal circle given by the pointer is the angle subtended by the two objects, and in like manner the vertical angle between two objects may be determined.

The above description will serve to explain the principle of the theodolite, although it does not exactly correspond with the details of the instrument. Thus, the graduation is described as being read off by a simple pointer, whereas in fact an instrument called a vernier, which we shall presently describe, is used, for the sake of obtaining greater accuracy. The movable horizontal circle, instead of being graduated all round, as above stated, is provided with a vernier, the graduation being on the base of the instrument itself, on which the circle turns. A clear idea of the details of the instrument and of the precautions necessary to insure accuracy in its use, cannot be easily obtained without actual practice.

63. *The Vernier* is used to secure greater accuracy in reading off the graduated limbs of other instruments. When a graduated rim moves by the side of a fixed mark or pointer through a number of complete graduations, the exact space through which it has moved may be read off, but it must generally happen in practice that the space to be estimated consists of a certain number of complete

graduations and the fraction of another, the exact size of which cannot be given by the pointer. Thus if the rim of a circle is graduated at intervals corresponding to an angle of 20″, the pointer will tell us the multiple of 20″ contained by an observed angle, leaving the quantity by which it exceeds such a multiple uncertain. This might be remedied by a closer graduation, but that is found to be attended with great practical difficulties, and the following method is employed.

CD is a portion of a graduated rim, *AB* a rim in contact with it, called a vernier, so graduated that



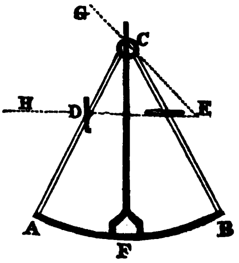
20 divisions of the vernier are equal in length to 19 of the rim. The difference between an interval of the rim and one of the vernier is therefore equal to $\frac{1}{20}$th of the former. Considering the zero point of the vernier as a pointer, let us suppose it to stand between two divisions of the graduated rim, and suppose, for example, that the graduation of the vernier marked 8 exactly corresponds with one of the graduations of the rim. Then the distance of this latter graduation from C=8 intervals of the rim, and its distance from the pointer =8 intervals

of the vernier, therefore the distance between the graduation *C* of the rim and the pointer equals 8 times the difference between an interval of the rim and an interval of the vernier, which is $\frac{1}{20}$th of an interval of the rim. In the case supposed, therefore, the fractional part of an interval requiring to be read off is equal to $\frac{8}{20}$ths of an interval, or if the intervals equal 20″, to 8″. So in any other position of the vernier, the value of the corresponding fraction will be indicated by the graduation of the vernier, which coincides with one of the lines on the rim.

If none of the graduations exactly coincide, those which most nearly do so must be treated as coincident.

We are thus enabled to observe accurately to a single second with an instrument graduated only at intervals of 20″.

64. *The Sextant* is used for measuring angles in any plane. Its construction is as follows. *AB* is a graduated circular arc, on which every degree is marked as two, so that the angle read off is double of the angle subtended at its centre *C*. *CF* is an arm moveable about *C* and furnished with a vernier *F*. At *C* a small mirror, perpendicular to the plane *ACB*, is attached to the arm *CF* and moves



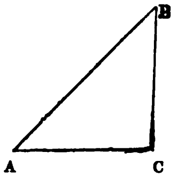
with it. Another mirror also perpendicular to the same plane, of which the upper half only is silvered and the lower left transparent is fixed at *D*. A telescope is placed at *E* and pointed to the division between the silvered and the transparent parts of the mirror *D*, so that an object reflected in the upper portion can be seen together with another object visible directly through the lower. The instrument is so adjusted, that when the zero of the vernier stands at the zero graduation of the arc, the image of an object formed by reflection in the two mirrors is seen through the telescope together with the object itself. To determine the angle between two objects *G* and *H*, the telescope is directed to one, as *H*, and the arm *CF*, with its accompanying mirror, is turned until the image of the other, after reflection in both mirrors, is also visible in the same direction. Then if *B* be the zero point of the arc, it can be shewn from optical principles that the angle *GEH* between the two objects is double the angle *BCF*, and is therefore equal to the angle read off from the arc. The instrument is called a sextant because the angle *ACB* is about 60°. The angles which the instrument can measure therefore extend to 120°. Similar instruments, in which *ACB* is 45°, are called quadrants.

65. By means of these instruments we can measure the distance between any accessible points and

the angles subtended by any visible objects. We can also, by means of the theodolite, find the angle which the direction in which an object is seen makes with the horizontal plane, the telescope being directed horizontally when the reading of the vertical circle is zero. This angle is called the angle of elevation or depression of the object according as it is above or below the horizontal plane. By applying the results of Chap. IV. to such observations, we may discover the values of distances and angles which are not susceptible of immediate measurement, as in the following examples:

66. *Examples of the determination of heights, distances, &c.*

1. To determine the height of a vertical object standing on a horizontal plane.

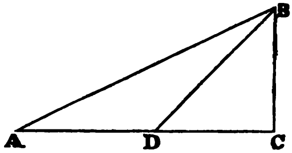


Let BC be the object. From its base C measure a line AC in the horizontal plane, and measure the angle CAB. Then

$$BC = AC \,.\, \tan BAC$$
$$\therefore \quad \log BC = \log AC + L \tan BAC - 10$$

The length of AC should be sufficient to prevent the angle BAC from being too near to 90°.

2. To determine the elevation of an object above a horizontal plane when the base is inaccessible.



Let CB be the object, ADC the horizontal plane. BC vertical, and C by the hypothesis inaccessible. Measure the distance AD between two points lying in the same straight line with C and observe the angles BAC, BDC. Then $ABD = BDC - BAD$. Then in the triangle ABD we have AD and the angles BAD and ABD, from which data BD may be determined as in Art. 57.

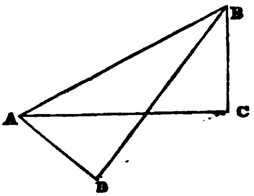
Again, in the right-angled triangle BDC we have BD and the angle BDC, from which the required value of BC may be found. In this way it will be found that

$$\log BC = \log AD + L \sin BAD + L \sin BDC - L \sin ABD - 10$$

3. To determine the elevation of an object standing upon any base on a horizontal plane.

In this case the point C, where a vertical through the apex B meets the horizontal plane is not a visible point. In order, therefore, that A, D and C may be in the same straight line we must ascertain by means of the theodolite that A, B and D are in the same vertical plane. The angles BAC, BDC being the angles of elevation of B from A and D, may also be measured, and we can then proceed as in the last example.

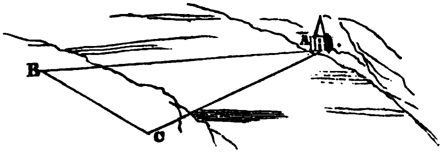
4. To determine the height of any object above an accessible point not situated in a horizontal plane.



Let B be the object, A the point above which its height is required. Draw AC horizontal, and BC vertical. Measure a base AD in any direction, and from A observe the angle BAD and the angular elevation of B, which will be the angle BAC. From D observe the angle BDA. Then in the triangle BAD we have the base AD and the two adjacent angles BAD and BDA, whence we may determine the side AB.

Again, in the right-angled triangle BAC we have the side AB and the angle BAC, from which data the required side BC may be determined. Thus we find

$$\log BC = \log AD + L \sin ADB + L \sin BAC - L \sin (BAD + ADB) - 10$$

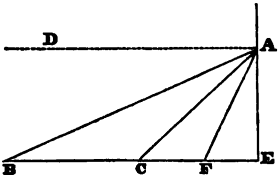


5. To determine the breadth of a river.

Measure a base BC near the side of the river, and from *B* and *C* measure the angles *ABC* and *ACB*, *A* being a conspicuous point on the opposite bank. Then, in the triangle *ABC* the base *BC* and the two adjacent angles are known, whence we may find the side *AB* or the breadth of the river. Thus we find

$$\log AB = \log BC + L \sin ACB - L \sin(ABC + ACB)$$

6. To determine the breadth of a river by observations from a neighbouring elevation of known height.



Let *A* be the point of observation, *BC* the breadth of the river required. From *A* observe the angles of depression of *B* and *C*, that is, the angles *DAB*, *DAC*; *DA* being horizontal. These angles are respectively equal to *ABC* and *ACE*, (*BCE* being horizontal). Then, in the right-angled triangle *ACE* we have the side *AE* and the angle *ACE*, whence *AC* may be found. Again, in the triangle *ABC* we have the side *AC* and the two angles *B* and *C*, whence *BC* may be determined. Thus we find

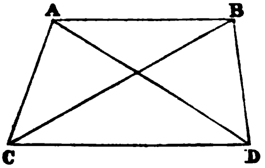
$$\log BC = \log AE + L \sin (ACE - ABC) - \\ - L \sin ABC - L \sin ACE + 10$$

7. *BC* may also be determined without knowing the height of *A*, if the distance from *C* of any point *F* in *BC* produced is known. For by observing in

addition to the former angles the depression of the point *F*, we obtain data for the determination of *AC* from the triangle *ACF*, and thence of *BC*, as before. Thus we find

$$\log BC = \log CF + L \sin (ACE - ABE) +$$
$$+ L \sin (AFE - ABE) - L \sin ABE - L \sin AFE$$

8. To determine the distance between two objects, both of which are inaccessible.



Let *A* and *B* be the two objects. Measure a base *CD* and observe the angles *ACD*, *BCD*, *ADC* and *BDC*. Then the angles *CAD*, *CBD* and *ADB* are also known, and from the triangle *ACD* we may determine *AD*, and from *BCD*, *BD*. Then the two sides *AD*, *BD* and the included angle *ADB* being known, *AB* may be found.

Thus we obtain

$$\log AD = \log CD + L \sin ACD - L \sin DAC$$
$$\log BD = \log CD + L \sin BCD - L \sin CBD$$

whence *AD* and *BD* are known

and $$L \tan \phi = \log 2 + \frac{1}{2} \log AD + \frac{1}{2} \log BD - \frac{1}{2} \log (AD + BD) + L \sin \frac{ADB}{2}$$

and $$\log AB = \log (AD - BD) + L \sec \phi - 10$$

9. A, B, C are three stations, and from B a base $BD(a)$ is measured along BA. The observed values of the angles CBD BDC BCA are α, β, γ. The depression of A below C and the elevation of B above C are also observed to be equal to δ, ϵ respectively. Find the distances of A, B, C from each other and the heights of B and C above A.

$$\log BC = \log a + L \sin \beta - L \sin (\alpha + \beta)$$
$$\log AC = \log BC + L \sin \alpha - L \sin (\alpha + \gamma)$$
$$\log AB = \log BC + L \sin \gamma - L \sin (\alpha + \gamma)$$

And if h be the height of C above A
h' be the height of B above C

$$\log h = \log AC + L \sin \delta - 10$$
$$\log h' = \log BC + L \sin \epsilon - 10$$

and height of B above $A = h + h'$.

10. A person wishing to determine the length of an inaccessible wall, places himself due south of one end and then due west of the other, at such distances that the angles subtended by the wall at the two positions each equal α. The distance between the two positions is a. Shew that the length of the wall $= a . \tan \alpha$.

11. A person standing at a known distance from two towers in the same straight line, observes their apparent altitudes to be the same. He then walks a given distance on level ground towards the towers,

till the angle of elevation of one is double that of the other. Find the heights of the towers.

Let the distance of the towers from the observer, at the first observation be a, b, and let c be the distance walked, h, h' the heights of the towers.

$$\log h = \frac{1}{2}\log a + \frac{1}{2}\log (b-c) + \frac{1}{2}\log (ab+ac-2bc) - \log b$$

$$\log h' = \frac{1}{2}\log a + \frac{1}{2}\log (b-c) + \frac{1}{2}\log (ab+ac-2bc) - \log a$$

12. A person standing on a mountain a feet high, takes the angles of depression α, β of the foot and summit of another conical mountain, and observes a person who starts to ascend the latter in the vertical plane through the two summits. After t hours he observes the angle of depression (γ) of the man's position. Find the mountain's height and its inclination, the man's velocity being b feet an hour.

Let θ be the mountain's inclination, h its height.

Then

$$L \sin (\theta+\gamma) = \log a - \log t - \log b + L \sin (\alpha-\gamma) - - L \sin \alpha + 10$$

$$\log h = \log a + L \sin \theta - L \sin (\theta+\beta) + + L \sin (\alpha-\beta) - L \sin \alpha$$

13. On the side of a hill there are two places

B, *C* known to be each at a distance d from *A*; at the lower place *C* the angular separation α of *A* and *B* is measured and also their altitudes λ, μ. Shew that

$$BC=2\,d\left\{\cos(\lambda-\mu)\cos^2\frac{\alpha}{2}-\cos(\lambda+\mu)\sin^2\frac{\alpha}{2}\right\}$$

and adapt the expression to logarithmic calculation.

14. *A*, *B*, *C* are three inaccessible objects. *D* and *E* two stations in *AB* and *AC* produced, whose distance equals a. The observed values of *BDE*, *CDE*, are α, β and of *BED*, *CED*, γ, δ respectively. Determine the distances between *A*, *B* and *C*.

$$\log AB=\log a+L\sin\alpha+L\sin(\delta-\gamma)-\\ -L\sin(\alpha+\gamma)-L\sin(\alpha+\delta)$$

$$\log AC=\log a+L\sin\delta+L\sin(\alpha-\beta)-\\ -L\sin(\delta+\beta)-L\sin(\alpha+\delta)$$

and *BC* may then be found from the triangle *ABC*, of which two sides and the included angle are known.

15. The shadows of two vertical walls at right angles to one another and a, a_1 feet high respectively, are observed, when the sun is due south, to be b and b_1 feet broad. Shew that if α be the sun's altitude and β the inclination of the first wall to the meridian,

$$\cot\alpha=\sqrt{\frac{b^2}{a^2}+\frac{b_1^{\,2}}{a_1^{\,2}}}\qquad \cot\beta=\frac{ab_1}{a_1b}$$

and adapt the former expression to logarithmic calculation.

The meridian is a vertical plane due north and south.

16. Two objects B and C lie in the same meridian at a distance a. Their bearings are observed from a station A and the angles subtended by AB and AC from another station D are observed to be γ and δ. Find the bearings of B and C from D and the distance AD.

By the bearing of an object is meant the inclination to the meridian of a vertical plane through the object and the point of observation.

Let the observed bearings of B and C from A be α, β
Let the required bearings of B and C from D be θ, ϕ

Then $$\theta-\phi=\delta-\gamma$$

$$L\tan\frac{\theta+\phi-\alpha-\beta}{2}=\log(m+1)-\log(m-1)+L\tan\frac{\alpha-\beta+\gamma-\delta}{2}$$

where $\log m=L\sin\delta+L\sin\beta-L\sin\gamma-L\sin\alpha$

These equations determine θ and ϕ.

And $$\log AD=\log a+L\sin\alpha+L\sin(\phi-\beta)--L\sin\delta-L\sin(\alpha-\beta)$$

The principal steps in the proof are first to determine AB and AC, then to express AD in terms of

ϕ from the triangle *ACD* and in terms of θ from the triangle *ABD*, and equate the results, which will give the above equation for determining $(\theta+\phi)$.

17. The distances of three points *A*, *B*, *C* from a station at which *AB*, *AC* and *BC* subtend equal angles are *a*, *b*, *c* respectively. Shew that

$$AB^2=a^2+b^2+ab$$
$$AC^2=a^2+c^2+ac$$
$$BC^2=b^2+c^2+bc$$

and adapt these equations to logarithmic calculation.

18. A person walking along a straight road *CE* can see the summit *A* of a hill except at *E*, where it is just hidden by the summit *B* of another hill. He measures *EC* (a) and at *E* observes the angle *CEA* (γ), at *C* he observes the angles *ACE* (α) and *BCE* (β) and the altitude (δ) of the summit *A*. Find the heights (h, h') of the hills *A*, *B*.

$$\log h=\log a+L\sin\gamma+L\sin\delta-h\sin(\alpha+\gamma)-10$$
$$\log h'=\log h+L\sin\beta+L\sin(\alpha+\gamma)-L\sin\alpha-$$
$$-L\sin(\beta+\gamma)$$

19. *ABCD* is a square, *E* a point in *CD* produced. A person holding *EC* horizontal and keeping the plane of the square vertical, sees the top of a tree in the direction *EB*, and after approaching *a* feet nearer to the tree sees the top in the direction *EA*. Shew that the tree is *a* feet high.

20. A person walking along a straight road observes the elevation (α) of a tower, whose shortest distance from the road $=a$. He also observes the angle (β) between the top of the tower and an object in the road. Find the height of the tower.

Let AB be the tower, C the nearest point of the road, P the point from which the observations are taken. Then from the three right-angled triangles ABC, PBC, PBA, we obtain three equations connecting AB, PB and BC, whence

$$\log AB = \log a + L \sin \alpha - L \sin \beta - L \cos \phi + 10$$

$$\text{where} \qquad L \sin \phi = L \sin \alpha - L \sin \beta + 10$$

21. A boy was flying a kite, at noon, when the wind was blowing from a direction α from the south, and the angular distance of the kite's shadow from the north was β. The wind suddenly changed to α' from the south and the shadow to β' from the north, and the kite rose as much above 45° as it had been below it. Shew that if θ was the sun's altitude and $45° - \phi$ that of the kite at first.

$$L \tan \theta = \frac{1}{2}\Big\{L \sin \beta + L \sin \beta' - L \sin (\alpha - \beta) - \\ - L \sin (\alpha' - \beta')\Big\} + 10$$

$$L \tan (45 - \phi) = \frac{1}{2}\Big\{L \sin (\alpha - \beta) + L \sin \beta' - \\ - L \sin (\alpha' - \beta) - L \sin \beta\Big\} + 10$$

22. A vessel observed another, α from the north, sailing in a direction parallel to its own. After p hours, its bearing was reduced to β, and after q hours more, to γ from the north. To what point of the compass were the vessels steering?

Let the required direction be θ from the north. Then

$$L \tan (\theta-\alpha)=L \tan (\alpha-\gamma)-2\, L \tan \phi+20$$

where

$$L \sec \phi=\frac{1}{2}\cdot\Big\{\log p-\log q+L \sin (\beta-\gamma)-$$
$$-L \sin (\alpha-\beta)-L \cos (\alpha-\gamma)\Big\}+15$$

23. A base AB, 300 feet in length, is measured on a level plain, in a direction pointing to the base of a tower. The angles of elevation of the tower from A and B respectively are 34° 20′ 9″ and 47° 19′ 10″. Find the height of the tower.

Using the figure of Ex. 2, we have

$AD=300 \quad BAD=34^\circ\, 20'\, 9'' \quad BDC=47^\circ\, 19'\, 10''$

$\therefore\ ABD=47^\circ\, 19'\, 10''-34^\circ\, 20'\, 9''=12^\circ\, 59'\, 1''$

and $BD=AD\cdot\dfrac{\sin BAD}{\sin ABD}$

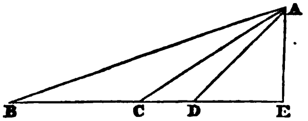
$$\therefore \log BD=\log 300+L \sin 34^\circ\, 20'\, 9''-$$
$$-L \sin 12^\circ\, 59'\, 1''$$

and $BC = BD \sin 47^\circ\, 19'\, 10''$

$$\therefore \log BC = \log 300 + L \sin 34^\circ\, 20'\, 9'' - $$
$$- L \sin 12^\circ\, 59'\, 1'' + L \sin 47^\circ\, 19'\, 10'' - 10$$
$$= 2.47712 + 9.75131 - 9.35155 +$$
$$+ 9.86637 - 10$$
$$= 2.74325$$
$$\therefore BC = 553.67$$
$$= 553 \text{ ft. } 8 \text{ in.}$$

24. From the top of a hill the angles of depression of the two banks of a river and of a buoy situated 100 yards from the nearer shore are observed to be 21° 9′, 45° and 32° 12′ 30″ respectively, all the observed points being in the same vertical plane with the top of the tower. Determine the breadth of the river.

Let BD be the two banks, C the buoy, then $CD = 100$ yards; A the point of observation.



Then $ABC = 21^\circ\, 9'$ $ACD = 32^\circ\, 12'\, 30''$ $ADE = 45^\circ$

$$\therefore \quad AD = CD \cdot \frac{\sin ACD}{\sin CAD}$$

$$\therefore \log AD = \log 100 + L \sin 32^\circ\, 12'\, 30'' -$$
$$- L \sin 12^\circ\, 47'\, 30''$$

Also $\quad BD = AD \dfrac{\sin BAD}{\sin ABD}$

$$\begin{aligned}\therefore\ \log BD &= \log AD + L\sin 23^\circ\ 51' - L\sin 21^\circ\ 9' \\ &= 2 + L\sin 32^\circ\ 12'\ 30'' + L\sin 23^\circ\ 51' \\ &\qquad - L\sin 12^\circ\ 47'\ 30'' - L\sin 21^\circ\ 9' \\ &= 2 + 9.72673 + 9.60675 \\ &\qquad - 9.34519 - 9.55728 \\ &= 2.43101\end{aligned}$$

$$\therefore\ BD = 269.78 = 269 \text{ yds. } 2 \text{ ft. } 4 \text{ in.}$$

25. The angle of elevation of a wall observed from a point 120 feet distant from its foot is 30° 15′. Required the height of the wall and the length of a scaling-ladder that will reach from the place of observation to its summit.

Ans. Height = 70 ft. Length = 138 ft. 11 in.

26. The angle of elevation of a steeple is observed to be 39° 30′, and 145 feet farther off it is 34° 15′. Find the height of the steeple. *Ans.* 567 ft. 3 in.

27. From the top of a hill the angles of depression of the top and bottom of a steeple, whose height is known to be 120 feet, were observed to be 30° and 33° respectively. Find the height of the hill. *Ans.* 1081 ft. 6 in.

28. A cape and a headland are observed from a ship to bear N.W. by W. and N.N.E. respectively, and after sailing 21 miles E.N.E ½ E. their bearings are W.N.W. and N. by W. ½ W. respectively. Find the distance between the cape and the headland.

Ans. 28 miles 149 yds.

29. At a distance of 100 feet from the base of a tower standing on level ground the angular elevation of the top is observed to be 51° 27′ 35″. Find the height of the tower. *Ans.* 125 ft. 6 in.

30. From the extremity *A* of a measured base of 1000 feet, the angle subtended by the top of a mountain and the other extremity *B* of the base is found to be 75° 13′ 27″ and the angular elevation of the mountain is 30° 24′ 30″. From *B* the angle subtended by *A* and the top of the mountain is 78° 10′. Find the height of the mountain above *A*. *Ans.* 1106 ft.

31. A base *AB*, of 300 feet in length, is measured along the side of a river, and an angle subtended at *A* by *B* and a post on the opposite bank is 81° 35′ 20″, and that subtended at *B* by *A* and the same post is 86° 24′ 15″. Find the distance of the post from *A*. *Ans.* 1439 ft. 2 in.

32. From a station *B* at the base of a mountain its summit *A* is seen at an elevation of 60°. After walking one mile towards the summit up a plane inclined 30° to the horizon to another station *C* the angle *BCA* is observed to be 135°. Find the height of the mountain in yards without employing logarithmic tables.

$$\begin{array}{c}\text{Height in}\\ \text{yards}\end{array} = 1760 \times \frac{\sin 60^\circ \sin 45}{\sin 15^\circ} = 1760\frac{3+\sqrt{3}}{2}$$

$$= 4164 \text{ yds. } 0 \text{ ft. } 5 \text{ in.}$$

33. A person standing on the edge of a river observes the angle of elevation of a tower on the opposite bank to be 55°. After receding 30 feet from the bank, he finds the angle of elevation of the tower to be 48°. Determine the breadth of the river. *Ans.* 104 ft. 11 in.

34. Two persons observe from a balloon that the angles of depression of two towns (one of which is due south of the other and distant from it $2\frac{1}{4}$ miles), are 45° and 60° respectively. The balloon then proceeds horizontally for 15 minutes, at the rate of 24 miles an hour, in a direction due S.E. and the angles of depression are then found to be exactly half their former values. Shew that the height of the balloon was 3 miles, very nearly.

35. A staff, on the top of a tower 200 feet high, subtends an angle of $6' 51''$ from a point in the horizontal plane 100 feet from the base of the tower. Shew that the height of the staff is one foot.

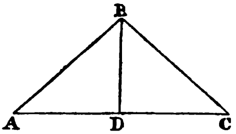
36. Two objects A and B were observed to be in a line inclined at an angle of 15° to the east of a ship's course, which was due north. The ship's course was then altered, and after sailing five miles N.W. the same objects bore E. and N.E. respectively. Shew that the distance $AB = \frac{10\sqrt{3}}{1+\sqrt{3}}$ miles, and find its value in miles and yards.

Other numerical examples may be made by giving particular numerical values to the data in Examples 1—22. The correctness of the results may be roughly tested by constructing the figures with mathematical instruments.

CHAPTER VI.

ON THE AREAS OF TRIANGLES AND POLYGONS, AND THE RADII OF THEIR INSCRIBED AND CIRCUMSCRIBED CIRCLES.

67. *To determine the area of a triangle.*



Let ABC be the triangle. Draw BD perpendicular to AC. Then the area of the triangle is half that of the rectangle contained by AC and BD. Euc. I. 41.

$$\therefore \quad \text{area} = \frac{1}{2} . AC . BD$$

$$= \frac{1}{2} bc \sin A$$

In the same way we might have obtained for the area the equivalent expressions $\frac{1}{2} ac \sin B$ and $\frac{1}{2} . ab . \sin C$.

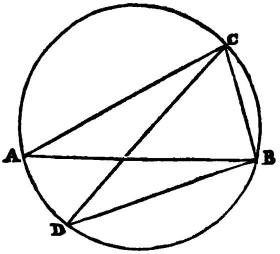
By Art. 57 we have

$$\sin A=\frac{2}{b\,.\,c}\sqrt{s\,.\,s-a\,.\,s-b\,.\,s-c}$$

$$\therefore\quad \text{area}=\sqrt{s\,.\,s-a\,.\,s-b\,.\,s-c}$$

which gives the area in terms of the sides.

68. *To find the radius of the circle described about a triangle.*



Let ABC be the triangle, CD a diameter of the circumscribed circle. Join BD. Then the angle CBD is a right angle. Euc. III. 31, and $CDB=CAB$. Euc. III. 21.

$$\therefore\quad CB=CD\,.\sin CDB$$

$$=CD\,.\sin CAB$$

Or if R is the radius of the circumscribed circle

$$a=2R\sin A$$

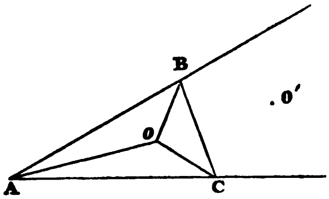
$$\therefore\quad R=\frac{a}{2\sin A}=\frac{b}{2\sin B}=\frac{c}{2\sin C}$$

Also since $\sin A=\frac{2}{b\,c}\sqrt{s\,.\,s-a\,.\,s-b\,.\,s-c}$

$$R=\frac{a\,b\,c}{4\sqrt{s\,.\,s-a\,.\,s-b\,.\,s-c}}$$

which gives R in terms of the sides.

69. *To find the radius of the circle inscribed in a triangle.*



Let O be the centre of the circle inscribed in ABC, r its radius. Then the perpendiculars from O on the sides of the triangle must all equal r.

Now

$$\text{area } ABC=\text{area } BOC+\text{area } COA+\text{area } AOB$$

$$\therefore\ \sqrt{s.s-a.s-b.s-c}=\frac{1}{2}ar+\frac{1}{2}br+\frac{1}{2}cr$$

$$=r.s$$

$$\therefore\ r=\sqrt{\frac{s-a.s-b.s-c}{s}}$$

The distances of the points of contact from the angles of the triangle may be found as follows:

Since the perpendiculars dropped from O upon the sides AB, AC are equal, and AO common to the two right-angled triangles so formed, these triangles are equal, and the angle A is bisected by AO. Hence the distance from A of each of the points of contact of the inscribed circle with AB or $AC=r\cot\frac{A}{2}$

$$=\sqrt{\frac{s-a.s-b.s-c}{s}}\ \sqrt{\frac{s.s-a}{s-b.s-c}}$$

$$=s-a$$

Similarly, the distances of the points of contact from B and C are $s-b$ and $s-c$ respectively.

Again, let r' be the radius of the circle described between BC and AB and AC produced, O' its centre. Then, if $O'A$, $O'B$, $O'C$ are joined, it is evident that

$$\text{area } ABC = \text{area } AO'B + \text{area } AO'C - \text{area } BO'C$$

$$\therefore\ \sqrt{s.s-a.s-b.s-c} = \frac{1}{2}cr' + \frac{1}{2}br' - \frac{1}{2}ar'$$

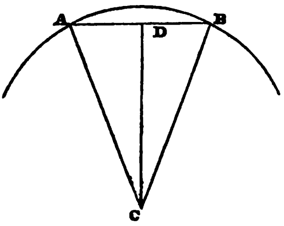
$$= r'(s-a)$$

$$\therefore\ r' = \sqrt{\frac{s.s-b.s-c}{s-a}}$$

and similar expressions may be found for the radii of circles touching the other sides externally.

70. *To find the area of a regular polygon and the radii of the inscribed and circumscribed circles.*

Let AB the side of a regular polygon of n sides $=a$. Let C be the common centre of the inscribed and circumscribed circles. Draw CD perpendicular to AB. Then CA and CD



are respectively the radii of the circumscribed and inscribed circles. Let $CA = R$, $CD = r$. Then since the polygon has n sides, the angle ACB must equal $\frac{1}{n}$th part of four right angles.

$$\therefore \quad ACD = \frac{180^\circ}{n}$$

$$\therefore \quad R = \frac{AD}{\sin ACD}$$

$$= \frac{a}{2 \sin \frac{180^\circ}{n}}$$

and $$r = AD \, . \cot ACD$$

$$= \frac{a}{2} \cot \frac{180^\circ}{n}$$

Also

$$\text{area of polygon} = n \, . \, \text{area } ACB$$

$$= \frac{1}{2} n \, . \, AB \, . \, CD$$

$$= \frac{1}{2} n \, a \, r$$

$$= \frac{1}{4} n \, a^2 \cot \frac{180^\circ}{n}$$

71. *Examples:*

1. If in any triangle ABC a straight line be drawn from C to the point D, where the inscribed

circle touches AB, and circles be inscribed in the two triangles so formed, they will touch CD in the same point.

For the distances from D at which the respective circles touch CD are $\frac{1}{2}(AD+CD-AC)$ and $\frac{1}{2}(BD+CD-BC)$ which are easily shewn to be the same, since $AD=\frac{1}{2}(b+c-a)$ and $BD=\frac{1}{2}(a+c-b)$.

2. If r be the radius of the inscribed circle of a triangle whose sides are a, b, c, prove that the product of the perpendiculars from the angles on the opposite sides $=\frac{(a+b+c)^3}{abc}r^3$.

This is easily proved by using the expressions for $\sin A$, $\sin B$ and $\sin C$ in terms of the sides.

3. In a right-angled triangle CD, CE are drawn from the right angle C, making angles α, β with the hypothenuse. Shew that the area of the triangle $CDE=\frac{a^2b^2}{2c^2}(\cot\alpha\sim\cot\beta)$.

4. On the sides of an equilateral triangle, three squares are described. Compare the area of the triangle formed by joining the centres of these squares with that of the equilateral triangle.

5. If R, r be the radii of the circles described

about and within the triangle ABC, the area of the triangle $= Rr(\sin A + \sin B + \sin C)$.

6. A circle is inscribed in a triangle, and a second triangle is formed whose sides are equal to the distances of the points of contact from the angles of the triangle. If r be the radius of the circle inscribed in the first triangle, and ρ, ρ' the radii of the inscribed and circumscribed circles of the second triangle, then will $r^2 = 2\rho\,\rho'$

7. The middle points of the sides of a triangle are joined with the opposite angles, and R_1, R_2, &c., are the radii of the circles described about the six triangles so formed, r_1, r_2, &c., those of the circles inscribed in the same; shew that

$$R_1 R_3 R_5 = R_2 R_4 R_6 \text{ and } \frac{1}{r_1} + \frac{1}{r_3} + \frac{1}{r_5} = \frac{1}{r_2} + \frac{1}{r_4} + \frac{1}{r_6}$$

8. A', B', C' are the angles subtended by the sides of the triangle ABC at the centre of the inscribed circle. Shew that

$$4 \sin A' \sin B' \sin C' = \sin A + \sin B + \sin C.$$

9. Four circles are described touching the sides of a triangle, or the sides produced. Shew that the circle which passes through any three of their centres has a radius double of that of the circle described about the triangle.

10. A circle is described about a triangle ABC, and a new triangle formed by joining the middle

points of the arcs subtended by the sides of ABC. Shew that the area of the new triangle

$$= \frac{\text{area of } A\,B\,C}{8 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}}$$

11. The area of a regular polygon of n sides inscribed in a circle is to the area of another of $3n$ sides in the same circle as $1 : k$. Shew that the angle which a side of the latter polygon subtends at the centre of the circle $= \cos^{-1} \frac{\sqrt{3+k}}{2k}$.

12. A line is drawn bisecting at right angles one side of a polygon of $2n+1$ sides. Shew that it passes through an angle of the polygon and that its length to that point $= \frac{a}{2} \cot \frac{180^\circ}{2(2n+1)}$, a being the length of a side of the polygon.

13. Three circles touch externally. Shew that their common tangents meet in a point distant $\sqrt{\frac{a\,b\,c}{a+b+c}}$ from each of the points of contact.

14. If a, a' be homologous sides of similar triangles described within and about a circle

$$a = 4\,a' \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$$

15. If R, r be the radii of circles described about

and within a regular polygon of n sides, shew that $R+r=\frac{a}{2}\cot\frac{90^\circ}{n}$, a being a side of the polygon.

16. The product of the radii of the four circles which touch the sides of a triangle, or the sides produced equals the square of the area of the triangle.

17. A pentagon, a hexagon and a decagon are inscribed in the same circle. Shew that a right-angled triangle may be formed whose sides are respectively equal to those of the three polygons.

THE END.

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